On the complexity of turning a graph into the analogue of a clique

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$$G \xmapsto{\text{orientation}} \overrightarrow{G}$$

Sometimes

$\overrightarrow{G}$ is o-clique

Does such orientation exist for a given graph?

How difficult is it to find the answer?

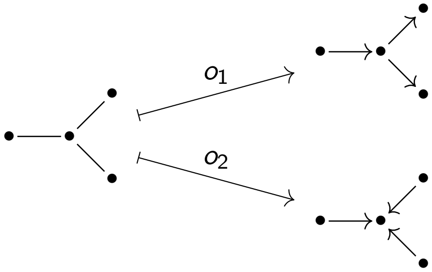
Outline

Definitions:

orientations, o-cliques, oriented distances and diameters, Klostermeyer-MacGillivray lemma

Orientation

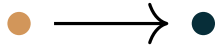
$G \xmapsto{\textbf{orientation}} \vec{G}$ is an assignment of a direction to each edge from undirected graph.



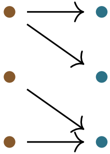
Oriented chromatic number

$\chi_o(\overrightarrow{G})$ is the minimal number of colors such that:

1. Colors of adjacent vertices are different



2. All arcs between two colors have the same direction

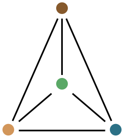


O-clique

oriented analogue of usual clique

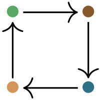
Undirected

G is clique $\iff \chi(G) = n$



Oriented

$\overrightarrow{G}$ is o-clique $\iff \chi_o(\overrightarrow{G}) = n$



Oriented distances and diameters

Strong

$$\mathrm{dist}_s = \max \begin{pmatrix} \mathrm{dist}(u, v) \\ \mathrm{dist}(v, u) \end{pmatrix}$$

$$\mathrm{diam}_s = \max_{u,v \in \overrightarrow{G}} \mathrm{dist}_s(u, v)$$

Weak

$$\mathrm{dist}_w = \min \begin{pmatrix} \mathrm{dist}(u, v) \\ \mathrm{dist}(v, u) \end{pmatrix}$$

$$\mathrm{diam}_w = \max_{u,v \in \overrightarrow{G}} \mathrm{dist}_w(u, v)$$

O-cliques & weak diameter

$\overrightarrow{G}$ is o-clique

$\Updownarrow$ $\Updownarrow$

$\chi_o(\overrightarrow{G}) = n \Longleftrightarrow \mathrm{diam}_w(\overrightarrow{G}) \leq 2$

[Klostermeyer, MacGillivray, 2004]

Decision problem:

Does G admit a 2-weak orientation?

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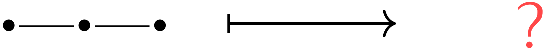
An orientation $G \mapsto \overrightarrow{G}$ is called 2-weak when $\mathrm{diam}_w(\overrightarrow{G}) \leq 2$



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$$\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet \quad \longmapsto \quad \bullet\!\longrightarrow\!\bullet\!\longrightarrow\!\bullet$$

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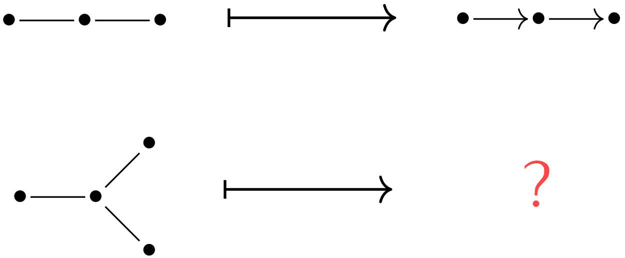

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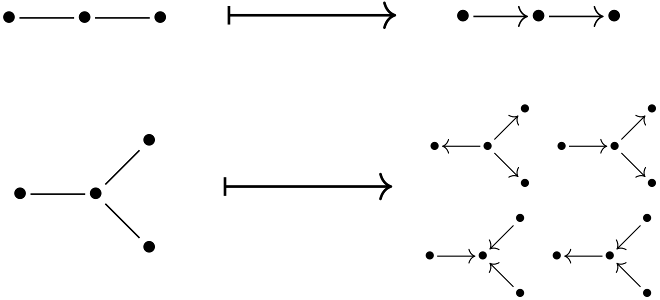

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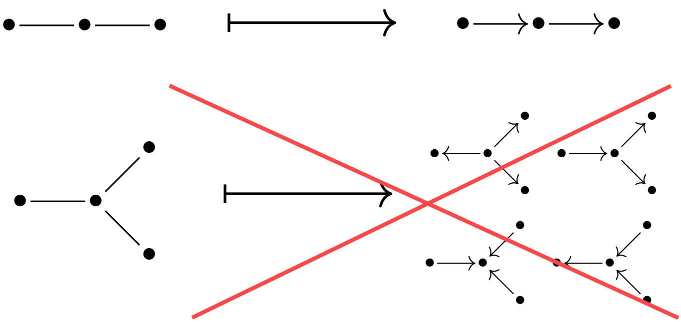

$\mathrm{diam}_w = \infty$ for any orientation of a claw

2-weak orientation is NP-complete

2-WEAK ORIENTATION is in NP

We just run BFS from all vertices to check whether $\text{diam}_w(\overrightarrow{G}) \leq 2$ or not.

2-WEAK ORIENTATION is NP-hard

We prove this by reduction from the monotone version of Not-All-Equal 3SAT

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

MONOTONE NOT-ALL-EQUAL 3SAT

F is 3CNF formula without negations.

Example: $(x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_3 \vee x_4) \wedge \ldots$

Is F satisfiable in such way that no clause have all literals set to same value?

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Is F satisfiable in such way that no clause have all literals set to same value?

NP-complete even when in every clause all variables are different

2-weak orientation is NP-complete

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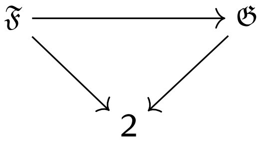
Reduction overview:



this diagram commutes

$\mathfrak{F}$ = nae-3-formulae

$\mathfrak{G}$ = graphs

$\mathbf{2} = \{\text{good, bad}\}$

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

$$F \mapsto G$$

$$F \text{ is nae-satisfiable } \iff \exists \overrightarrow{G} : diam_w(\overrightarrow{G}) \leq 2$$

$(\bullet, \bullet')$ is a representative pair from G

$(\bullet, \bullet')$ is a non-representative pair from G

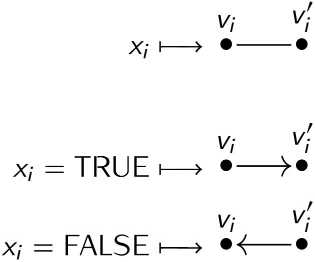
$\exists \overrightarrow{G} : \text{dist}_w(\bullet, \bullet') \leq 2 \iff F$ is nae-satisfiable

$\exists \overrightarrow{G} : \text{dist}_w(\bullet, \bullet') \leq 2$ by construction of G

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

Variable gadget:

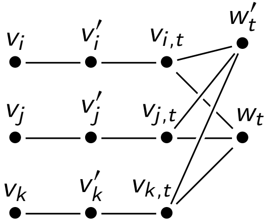


2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

Gadget for clause:

$C_t = x_i \vee x_j \vee x_k \longmapsto$



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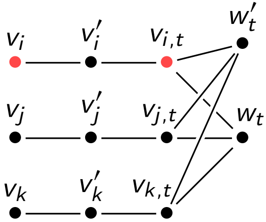

C_t is nae-satisfiable $\iff \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2$

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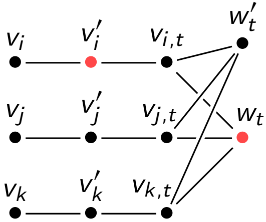

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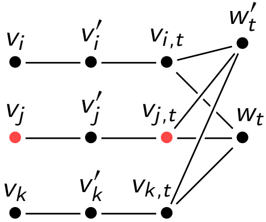

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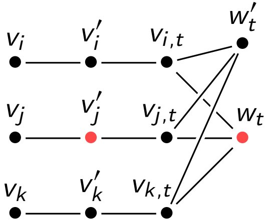

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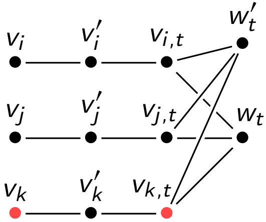

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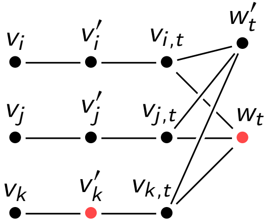

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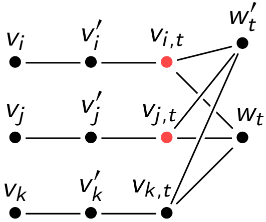

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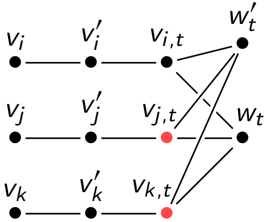

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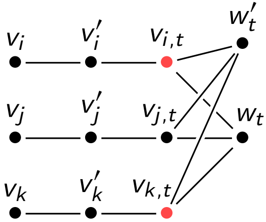

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2-weak orientation is NP-complete

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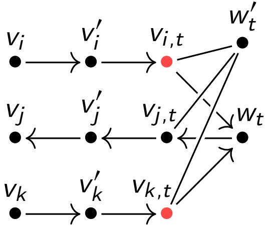


C_t is nae-satisfiable $\iff \exists \overrightarrow{G} : \text{dist}_w(\bullet, \bullet') \leq 2$

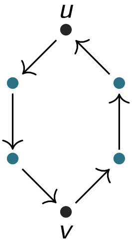
Using w_t' we can make $\text{dist}_w(\bullet, \bullet') \leq 2$ when it is needed

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

$\boldsymbol{G_F}$ is a union of clause gadgets. $G = G_F + \text{smth}$.

For any pair of vertices (u, v) from G_F we add the following:



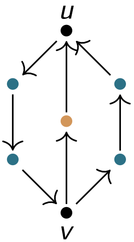
$\text{dist}_w(u, \bullet) \leq 2$
$\text{dist}_w(v, \bullet) \leq 2$

Here $\text{dist}_w(u, v) = 3$, so we don't break our reduction by adding blue vertices.

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

Also, for any **non-representative** pair of vertices (u,v) from G_F we add the following:



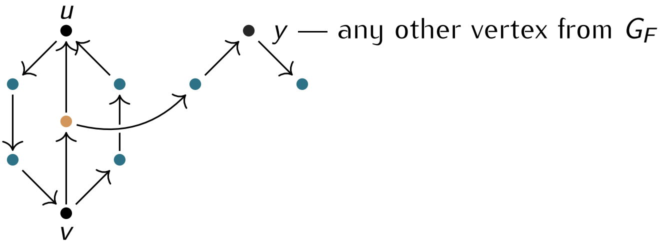
$\text{dist}_w(u, \bullet) = 1$
$\text{dist}_w(v, \bullet) = 1$

Here $\text{dist}_w(u, v) = 2$, so we ensure that the distance between non-representative vertices is ≤ 2

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

Next, in order to ensure that $\forall y \in G_F : \text{dist}_w(y, \bullet) \leq 2$ we add the following:



$\text{dist}_w(y, \bullet) = 2$

2-weak orientation is NP-complete

$(x_i \lor x_j \lor x_k) \land \ldots \land (x_i \lor x_k \lor x_l) \mapsto G$

Finally, turn all brown and blue vertices into one big clique,
by adding all possible edges between them.

2-weak orientation is NP-complete

$(x_i \vee x_j \vee x_k) \wedge \ldots \wedge (x_i \vee x_k \vee x_l) \mapsto G$

$$\exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \iff F \text{ is nae-satisfiable}$$

$$\left.\begin{array}{l} \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \\ \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \\ \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \\ \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \\ \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \\ \exists \overrightarrow{G} : \mathrm{dist}_w(\bullet, \bullet') \leq 2 \end{array}\right\} \text{by construction of } G$$

$$F \text{ is nae-satisfiable} \iff \exists \overrightarrow{G} : \mathit{diam}_w(\overrightarrow{G}) \leq 2$$

Family of related problems:

Does G admit an orientation $G \mapsto \overrightarrow{G}$ such that $\text{prop}(\overrightarrow{G})$?

Family of o-problems

Does G admit an orientation $G \mapsto \overrightarrow{G}$ such that prop($\overrightarrow{G}$)?

Strong orientation:

$\text{diam}_s(\overrightarrow{G}) < \infty \iff G$ is bridgeless [Robbins, 1939]

Linear time using Tarjan's bridge-finding algorithm [1974]

k-strong orientation:

$\text{diam}_s(\overrightarrow{G}) \leq k$

NP-complete for $k = 2$ [Chvátal, Thomassen, 1978]

Family of o-problems

Does G admit an orientation $G \mapsto \overrightarrow{G}$ such that prop($\overrightarrow{G}$)?

Weak orientation:

$$\mathrm{diam}_w(\overrightarrow{G}) < \infty \iff \text{B-contraction}(G) \text{ is claw-free.}$$

B-contraction means “replace all bridgeless subgraphs by vertices”

Linear time [Bensmail, Duvignau, Kirgizov 2013]

Family of o-problems

Does G admit an orientation $G \mapsto \overrightarrow{G}$ such that prop($\overrightarrow{G}$)?

<u>k-weak orientation:</u>

$\mathrm{diam}_w(\overrightarrow{G}) \leq k$

NP-complete for $k \geq 2$ [Bensmail, Duvignau, Kirgizov, 2013]

$\boldsymbol{G_k}$ is a k-edge-coloured graph
(e.g. 2-edge coloured graphs are just signified graphs)

$$\exists \overrightarrow{G} : \mathrm{diam}_w(\overrightarrow{G}) \leq k \iff \exists G_k : \chi_k(G_k) = n$$

Open question:

Complexity of k-strong orientation when $k > 2$

Thank you for your attention

kirgizov.complexnetworks.fr/some-problems/orientation.html