Pattern avoiding permutations modulo pure descents

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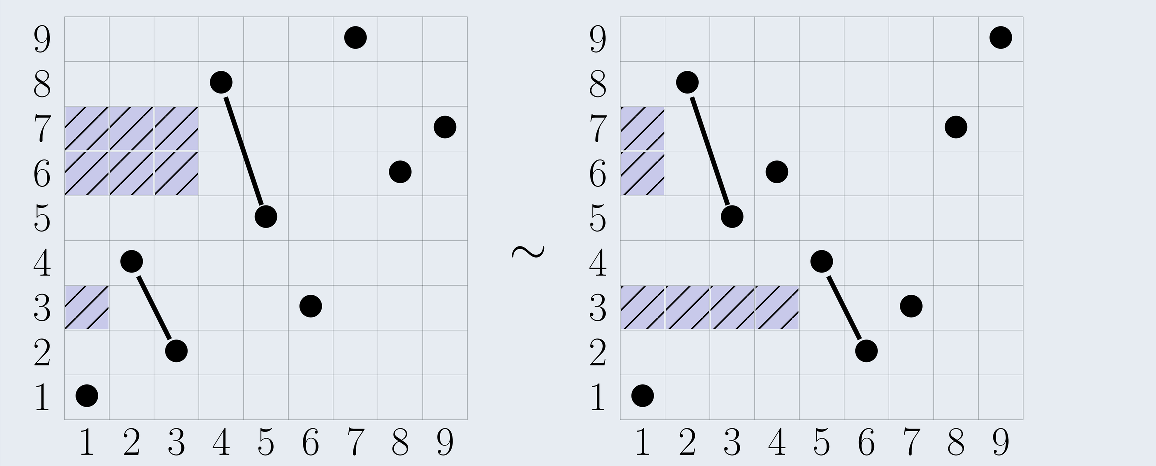
Pure descents in permutations

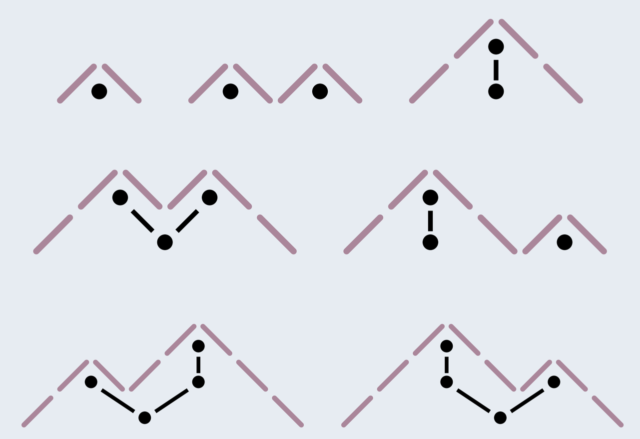


A descent $\pi_i\pi_{i+1}$ is pure if $\nexists j < i$ s.t. $\pi_{i+1} < \pi_j < \pi_i$. Equivalent permutations have the same values of pure descents.

- Regrouping values from consecutive pure descents we obtain a non-crossing partition. So, $S_n^\sim$ is enumerated by Catalan numbers.
- In $S_n(231)$ any descent is pure.
- $S_n(231)$ is a set of representative elements of $S_n^\sim$.

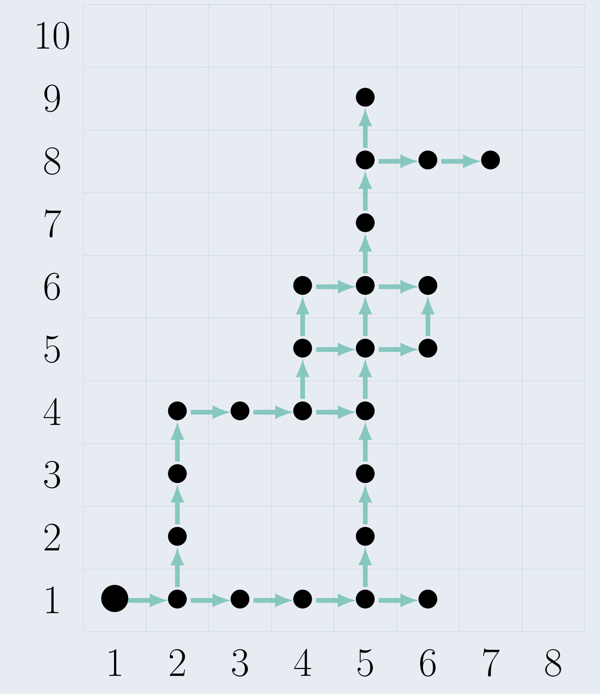
See also "The pure descent statistic on permutations" (by JLB and SK, to appear in DM), where authors prove, among other things, that the number of n-length permutations with k pure descents is given by the unsigned Stirling number of the first kind https://oeis.org/A132393. Thus, pure descents are equidistributed with cycles in permutations.

Forests



Ordered forests are enumerated by Catalan numbers. A trivial bijection sends such forests to Dyck paths.

Single-source directed animals

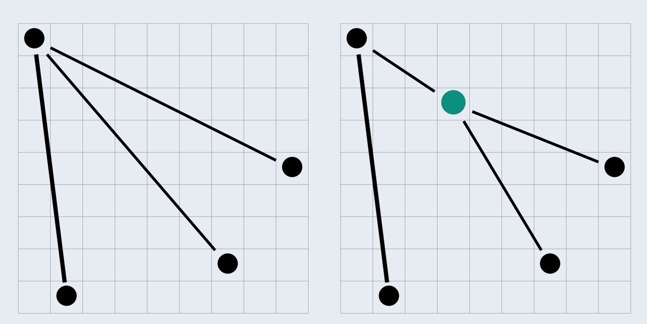


Single-source directed animal is a connected subset of $\mathbb{N}^2$ growing from $(0,0)$ by using $\uparrow$ and $\rightarrow$ steps.

"Directed animals, forests and permutations" (E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani) shows that single-source directed animals are in bijection with the forests of binary trees.

Barred pattern

$S_n(231, \underline{51}423)$ corresponds to a restricted $S_n(231)$ containing only binary trees. Thus, we obtain a subset of permutations enumerated by single-source directed animals.



Permutation avoids $\underline{51}423$ iff every $\underline{41}23$ pattern can be extended to $\underline{51}423$, where underlined entries are adjacent.

Number of equivalence classes for some restricted sets of pattern avoiding permutations

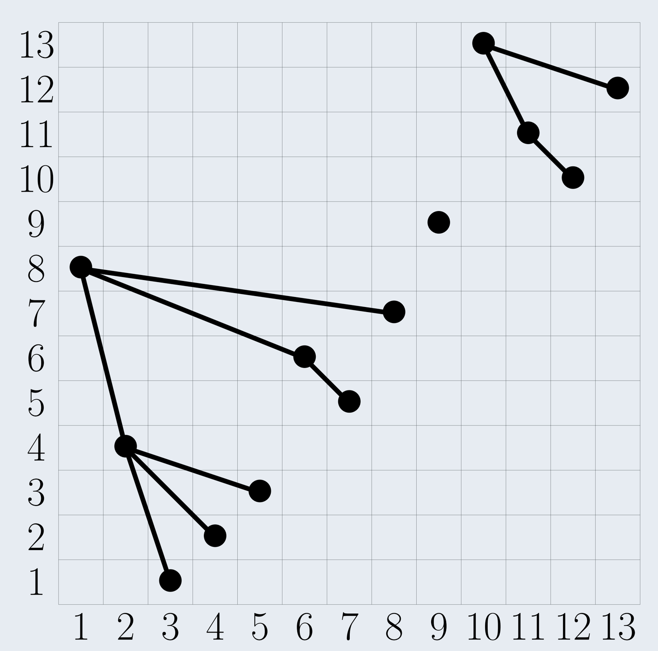
Pattern	Sequence	Sloane	a_n, $1 \le n \le 9$
$\{\}, \{231\}$	Catalan	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862
$\{312\}, \{321\}$	2^{n-1}	A011782	1, 2, 4, 8, 16, 32, 64, 128, 256
$\{231, \underline{51}423\}$	Directed animals	A005773	1, 2, 5, 13, 35, 96, 267, 750, 2123
$\{123\}$	Directed animals	A005773	1, 2, 5, 13, 35, 96, 267, 750, 2123
$\{132\}$	New		1, 2, 4, 10, 26, 66, 169, 437, 1130, ???
$\{213\}$	Special Dyck paths ?	A152225 ?	1, 2, 4, 9, 22, 56, 146, 388, 1048, ???

$S_n(312)^\sim$, $S_n(231)^\sim$

$S_n(312)^\sim = S_n(231, 312)$
$S_n(321)^\sim = S_n(231, 321)$

Using a result from "Restricted permutations" (R. Simion and F.W. Schmidt) we obtain 2^{n-1}.

$S_n(231)^\sim$ are in bijection with ordered forests



Permutation 8 4 1 2 3 6 5 7 9 13 11 10 12 with corresponding forest.

Our bijection transports the following statistics:

$S_n(231)$	des=ides	adj	lrM	rlm	inv	lmax	lsum
$\mathcal{F}_n$	ledg=redg	nod1	ordt	leav	vpat	dept	inpl

For a permutation π, we define:

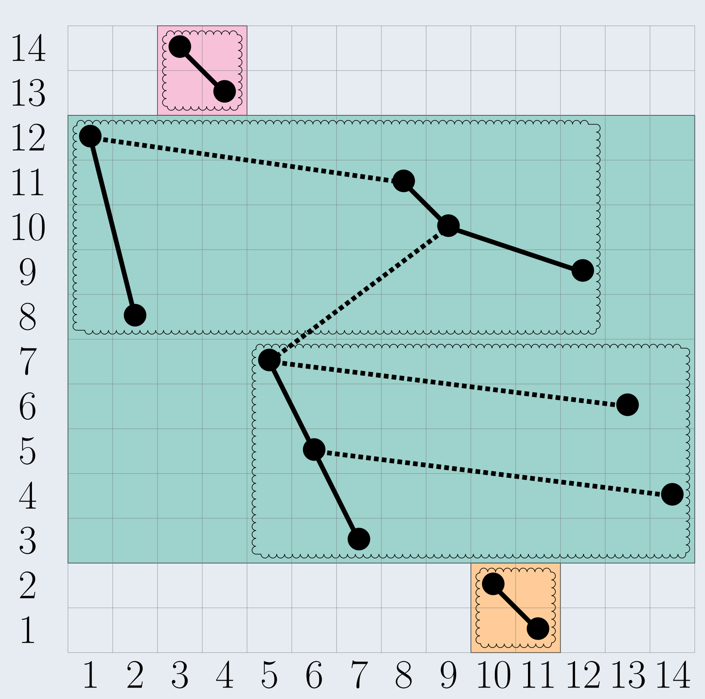
- **des**(π) = *number of descents* (which is also the number of pure descents);
- **ides**(π) = *number of descents in* π^{-1} (for $\pi \in S_n(231)$, we have **ides**(π) = **des**(π));
- **adj**(π) = *number of adjacencies, i.e.* descent (π_i, π_{i+1}) such that $\pi_{i+1} = \pi_i - 1$;
- **lrM**(π) = *number of left-to-right maxima, i.e.* $i \ge 1$ such that $\pi_i > \pi_j$ for all $j < i$;
- **rlm**(π) = *number of right-to-left minima, i.e.* $i \ge 1$ such that $\pi_i < \pi_j$ for all $j > i$;
- **inv**(π) = *number of inversions, i.e.* pairs (π_i, π_j) with $\pi_i > \pi_j$ and $i < j$,
- **lmax**(π) = *maximum value of the Lehmer code* $\ell_1\ell_2\ldots\ell_n$ of π, *i.e.* $\max_{1\le i\le n} \ell_i$ where $\ell_i = |\{\pi_j > \pi_i,\ j < i\}|$;
- **lsum**(π) = *sum of all values of the Lehmer code* of π.

For a forest $f \in \mathcal{F}_n$, we define

- **ledg**(f) = *number of left edges, i.e.*, leftmost edges among its siblings;
- **redg**(f) = *number of right edges, i.e.*, rightmost edges among its siblings (**ledg**(f)=**redg**(f));
- **nod1**(f) = *number of nodes with only one child*;
- **ordt**(f) = *number of ordered trees*;
- **leav**(f) = *number of leaves, i.e.*, nodes without child;
- **vpat**(f) = *number of vertical paths* (a vertical path is a path between a node and one of its ancestors);
- **dept**(f) = *depth, i.e.*, the maximal length of a vertical path;
- **inpl**(f) = *internal path length, i.e.*, the sum of the lengths of all paths from a node to the root.

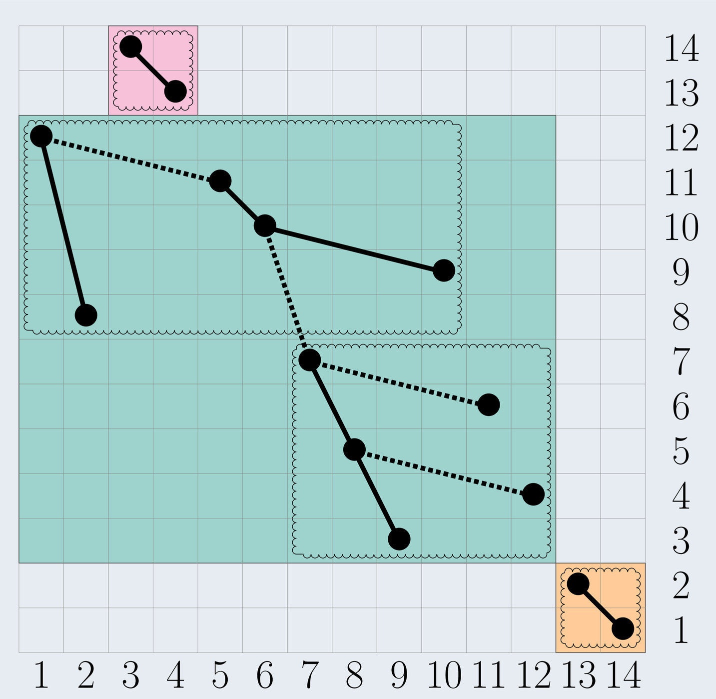
st	$S_n(231)$
	Catalan
des, ides, lmax	$\frac{1-z+zy-\sqrt{z^2y^2-2z^2y+z^2-2zy-2z+1}}{2zy}$
adj	$\frac{1-zy+z-\sqrt{z^2y^2+2z^2y-3z^2-2zy-2z+1}}{2z}$
lrM	$\frac{2}{2-y+y\sqrt{1-4z}}$
rlm	$\frac{1+z-zy-\sqrt{z^2y^2-2z^2y+z^2-2zy-2z+1}}{2z}$
inv, lsum	$F(z,y) = \frac{1}{1-z(F(zy,y)-1)-x}$

$S_n(123)^\sim$ are in bijection with directed animals



Left: A permutation from $S(123)^\sim$ together with corresponding ordered forest of binary trees, which is bijectively related to a single-source directed animal. The idea of the construction consists in linking consecutive pure descents and joining obtained *runs of pure descents* in certain order.

Right: The ordered forest of ordered binary trees, constructed from the permutation on the left side, corresponds to another permutation of the same class.



In the same class the blocks of pure descents moves only horizontally (indeed, it is the only way because two permutations from the same class, by definition, should have the same values of pure descents). Pushing the blocks "to the right" in certain way we obtain a representative permutation.

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