Enumeration of Dyck paths with air pockets

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Dyck paths with air pockets, $\mathcal{A}_n$

A *Dyck path with air pockets* is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geqslant 1$, where two down steps cannot be consecutive (we set $D = D_1$ for short). The set of such paths is denoted by $\mathcal{A}$.

Let $\mathcal{A}_n$ denote the set of n-length Dyck paths with air pockets.

Example:

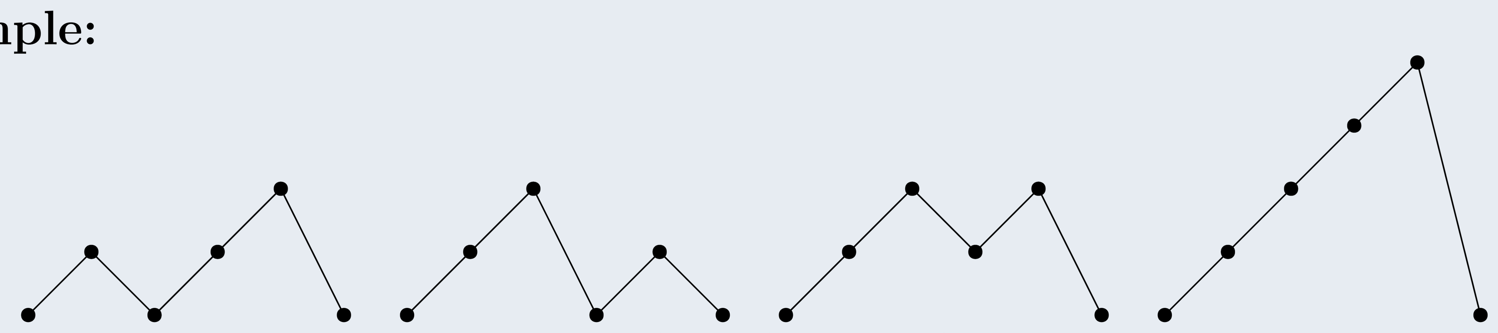


Figure: $\mathcal{A}_5 = \{UDU^2D_2, U^2D_2UD, U^2DUD_2, U^4D_4\}$.

Cardinality formula (OEIS A004148):

$$|\mathcal{A}_n| = \sum_{k=\lceil n/2\rceil}^{n} \frac{1}{k}\binom{k}{n-k}\binom{k}{n-1-k}.$$

Generating function:

$$\sum_{n=2}^{\infty} |\mathcal{A}_n| \cdot x^n = \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x}.$$

A Dyck path with air pockets is *prime* if it ends with D_k, $k \geqslant 2$, and it returns to the x-axis only once. The set of such paths is denoted by $\mathcal{P}$.

Lowered and elevated paths

We introduce two transformations of Dyck paths with air pockets. If α is a Dyck path with air pockets of the form $U\beta D_k$ (where β is either empty or in $\mathcal{A}$), then we define the *elevated* version of α as $\alpha^\sharp = U^2\beta D_{k+1}$. We also define the inverse operation $\cdot^\flat$, and call $\alpha^\flat$ the *lowered* version of α.

Example:

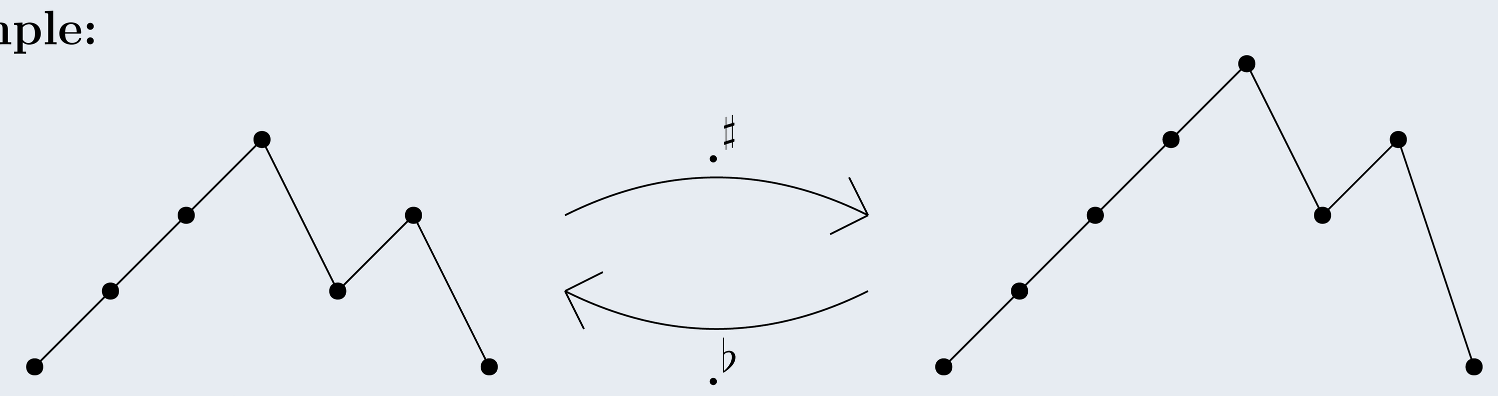


Figure: The paths $U^3D_2UD_2$ (left) and $U^4D_2UD_3$ (right) are lowered and elevated versions of one another.

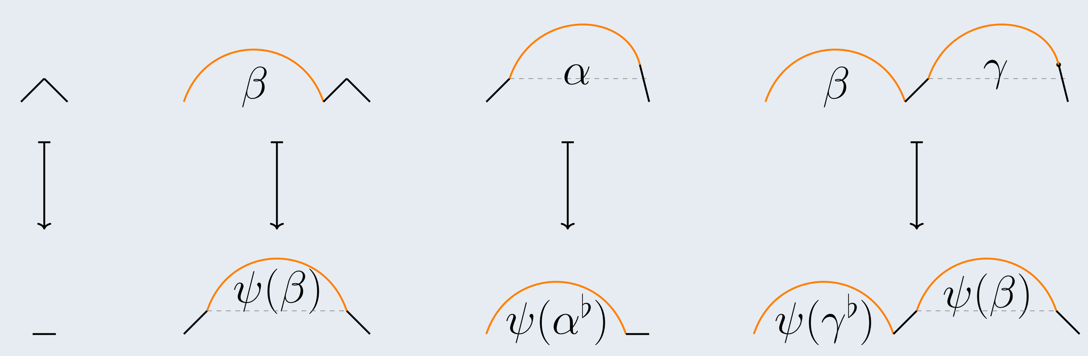
The operations $\cdot^\flat$ and $\cdot^\sharp$ will help us to define a bijection between $\mathcal{A}_n$ and a class of well-known lattice paths.

Bijection with peakless Motzkin paths, $\mathcal{M}_n$

A *peakless Motzkin path* is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, consisting of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and flat-steps $F = (1, 0)$, having no occurrence of UD. The set of such paths is denoted by $\mathcal{M}$, and the subset of n-length peakless Motzkin paths is denoted by $\mathcal{M}_n$.

$$\mathcal{A} \xrightarrow{\psi} \mathcal{M}$$

$$\alpha \overset{\psi}{\longmapsto} \begin{cases} F & \alpha = UD \\ U\psi(\beta)D & \alpha = \beta UD, \beta \in \mathcal{A} \\ \psi(\alpha^\flat)F & \alpha \in \mathcal{P} \\ \psi(\gamma^\flat)U\psi(\beta)D & \alpha = \beta\gamma, \beta \in \mathcal{A}, \gamma \in \mathcal{P} \end{cases}$$



Theorem: The map ψ induces a bijection between $\mathcal{A}_n$ and $\mathcal{M}_{n-1}$.

Pattern popularity generating function example

For all $k \geqslant 1$, the g.f. for the total number of occurrences of $U^\ell D_\ell$, $\ell \geqslant k$, in $\mathcal{A}_n$ is:

$$\frac{x^{k+1}\left(1 + 2x^2 - x^3 + (1-x)\sqrt{(x^2+x+1)(x^2-3x+1)}\right)}{2(1-x)\sqrt{(x^2+x+1)(x^2-3x+1)}}. \quad \text{(OEIS A201631)}$$

Transport of consecutive patterns

Dyck path with air pockets	Peakless Motzkin path
$\mathbf{U}$	$\mathbf{F + U = F + D}$
$\mathbf{D = UD}$	$\mathbb{1}_{\mathbf{F}} + \mathbf{UFD} + \mathbb{1}_{\mathbf{U\mathcal{M}D}} + \mathbf{U^2\mathcal{M}D^2}$
$\mathbf{DU}$	$\mathbf{UFD + U^2\mathcal{M}D^2}$
$\mathbf{UU}$	$\mathbf{F} - \hat{\mathbf{1}}$
$\mathbf{\Delta_k}$	$\mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}} + \mathbf{UF^{k-1}U\mathcal{M}D^2}$
Peak	$\mathbf{U} + \hat{\mathbf{1}}$
Ret	$\hat{\mathbf{n}} - \mathbf{LastF}$
SLast	**Ret**

For any path α (in $\mathcal{A}$ or in $\mathcal{M}$, depending on the relevant column in the table):

- $\hat{\mathbf{1}}(\alpha) = 1$, $\hat{\mathbf{2}}(\alpha) = 2$, $\hat{\mathbf{n}}(\alpha) = n$, and so on;
- $\mathbb{1}_{\boldsymbol{\beta}}(\alpha)$ is 1 if $\alpha = \beta$ and 0 otherwise;
- $\mathbb{1}_{\mathbf{U\mathcal{M}D}}(\alpha)$ is 1 if there exists $\beta \in \mathcal{M}$ such that $\alpha = U\beta D$ and 0 otherwise;
- $\mathbf{U^2\mathcal{M}D^2}(\alpha)$ is the number of occurrences of $U^2\beta D^2$ in α for $\beta \in \mathcal{M}$;
- $\mathbf{\Delta_k}(\alpha)$ is the number of occurrences of U^kD_k in α;
- $\mathbf{Peak}(\alpha) = \sum_{k \geqslant 1} \mathbf{UD_k}(\alpha)$;
- $\mathbf{Ret}(\alpha)$ is the number of returns to the x-axis of α;
- $\mathbf{LastF}(\alpha)$ is the position of the rightmost flat-step in α;
- $\mathbf{SLast}(\alpha)$ is the size of the the last step of α (i.e. k if the last step is D_k).

Pattern popularity in Dyck paths with air pockets

The popularity of various patterns (i.e. total number of occurrences of said patterns) in $\mathcal{A}_n$ ($2 \leqslant n \leqslant 11$) can be found in the following table:

Pattern	Pattern popularity in $\mathcal{A}_n$	OEIS
U	$1, 2, 5, 13, 32, 80, 201, 505, 1273, 3217$	A110320
D	$1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
Peak	$1, 1, 3, 7, 16, 39, 95, 233, 577, 1436$	A203611
Ret	$1, 1, 3, 6, 13, 29, 65, 148, 341, 793$	A093128
Cat	$0, 1, 1, 4, 8, 19, 44, 102, 239, 563$	
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
$\Delta_{\geqslant k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 30, 70, 167, 405$	A201631($= u_n$)
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ $1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	$u_n - u_{n-k}$
	$\Delta_{\leqslant 2}$ $1, 1, 2, 5, 10, 24, 47, 137, 335, 825, \ldots$	
	$\Delta_{\leqslant 3}$ $1, 1, 3, 5, 12, 27, 64, 154, 375, 922, \ldots$	
	etc.	

- $\mathrm{Cat}(\alpha)$ is the number of catastrophes of α, i.e. steps of the form D_k, $k \geqslant 2$, ending on the x-axis;
- $\Delta_{\geqslant k}(\alpha)$ is the number of occurrences of $U^\ell D_\ell$, $\ell \geqslant k$, in α;
- $\Delta_{\leqslant k}(\alpha)$ is the number of occurrences of $U^\ell D_\ell$, $1 \leqslant \ell \leqslant k$, in α.

We also define *non-decreasing Dyck paths with air pockets* as paths of $\mathcal{A}$ whose sequence of valley heights is non-decreasing, i.e. the sequence of the minimal ordinates of the occurrences of D_kU, $k \geqslant 1$, is non-decreasing from left to right. The set of such paths is denoted by $\mathcal{A}^\nearrow$, and the subset of n-length non-decreasing Dyck paths with air pockets is denoted by $\mathcal{A}_n^\nearrow$. We get the following table:

Pattern	Pattern popularity in $\mathcal{A}_n^\nearrow$	OEIS
U	$1, 2, 5, 13, 32, 76, 176, 400, 896, 1984$	A098156
D	$1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
Peak	$1, 1, 3, 7, 16, 36, 80, 176, 384, 832$	A045891
Ret	$1, 1, 3, 6, 13, 27, 56, 115, 235, 478$	A099036
Cat	$0, 1, 1, 4, 8, 18, 38, 80, 166, 342$	A175657
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
$\Delta_{\geqslant k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 28, 61, 133, 290, 631$	New ($= v_n$)
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ $1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	$v_n - v_{n-k}$
	$\Delta_{\leqslant 2}$ $1, 1, 2, 5, 10, 22, 48, 105, 229, 498$	
	$\Delta_{\leqslant 3}$ $1, 1, 3, 5, 12, 25, 55, 120, 262, 570$	
	etc.	

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