

Enumeration of Dyck paths with air pockets

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Abstract

We introduce and study the new combinatorial class of Dyck paths with air pockets. We exhibit a bijection with the peakless Motzkin paths which transports several pattern statistics and give bivariate generating functions for the distribution of patterns as peaks, returns and pyramids. Then, we deduce the popularities of these patterns and point out a link between the popularity of pyramids and a special kind of closed smooth self-overlapping curves, a subset of Fibonacci meanders. A similar study is conducted for the subclass of non-decreasing Dyck paths with air pockets.

Keywords: Dyck path, pattern distribution/popularity, Fibonacci meander

1 Introduction and notations

In combinatorics, lattice paths are widely studied. They have many applications in various domains such as computer science, biology and physics [21], and they have very tight links with other combinatorial objects such as directed animals, pattern avoiding permutations, bargraphs, RNA structures and so on [4, 11, 21]. A classical problem in combinatorics is the enumeration of these paths with respect to their length and other statistics [1, 2, 3, 7, 14, 15, 17, 18, 19]. In the literature, Dyck and Motzkin paths are the most often considered. They are counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in Sloane's On-line Encyclopedia

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of Integer Sequences [20]). In 2005, Dyck paths with catastrophes have been introduced by Krinik *et al.* in [12] in the context of queuing theory. They correspond to the evolution of a queue by allowing some resets. The push (resp. pop) operation corresponds to a step $U = (1, 1)$ (resp. $D = (1, -1)$), and the reset operations are modeled by catastrophe steps $D_k = (1, -k)$ ending on x -axis, $k \geq 2$. Banderier and Wallner [1] study these paths by providing enumerative results and limit laws.

In this paper, we introduce and study the paths with air pockets corresponding to a queue evolution with partial reset operations that cannot be consecutive. More formally, a *Dyck path with air pockets* is a non empty lattice path in the first quadrant of \mathbb{Z}^2 starting at the origin, ending on the x -axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geq 1$, where two down steps cannot be consecutive. For short, we set $D = D_1$. The *length* of a Dyck path with air pockets is the number of its steps. Let \mathcal{A}_n be the set of n -length Dyck paths with air pockets. By definition $\mathcal{A}_0 = \mathcal{A}_1 = \emptyset$ and we set $\mathcal{A} = \bigcup_{n \geq 2} \mathcal{A}_n$. A Dyck path with air pockets is called *prime* whenever it ends with D_k , $k \geq 2$, and returns to the x -axis only once. The set of all prime Dyck paths with air pockets of length n is denoted \mathcal{P}_n . Notice that UD is not prime so we set $\mathcal{P} = \bigcup_{n \geq 3} \mathcal{P}_n$. If $\alpha = U\beta UD_k \in \mathcal{P}_n$, then $2 \leq k < n$, β is a (possibly empty) prefix of a path in \mathcal{A} , and we define the Dyck path with air pockets $\alpha^b = \beta UD_{k-1}$, called the ‘lowering’ of α . For example, the path $\alpha = UUDUUD_3$ is prime, and $\alpha^b = UDUUD_2$. The map $\alpha \mapsto \alpha^b$ is clearly a bijection from \mathcal{P}_n to \mathcal{A}_{n-1} for all $n \geq 3$, and we denote by γ^\sharp the inverse image of $\gamma \in \mathcal{A}_{n-1}$ (α^\sharp is a kind of ‘elevation’ of α , drawing inspiration for the term from Deutsch’s definition of elevated Dyck paths [7]). Any Dyck path with air pockets $\alpha \in \mathcal{A}$ can be decomposed depending on its *second-to-last return to the x -axis*: either (i) $\alpha = UD$, or (ii) $\alpha = \beta UD$ with $\beta \in \mathcal{A}$, or (iii) $\alpha \in \mathcal{P}$, or (iv) $\alpha = \beta\gamma$ where $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{P}$. So, if $A(x) = \sum_{n \geq 2} a_n x^n$ where a_n is the cardinality of \mathcal{A}_n , and $P(x) = \sum_{n \geq 3} p_n x^n$ where p_n is the cardinality of \mathcal{P}_n , then we have $P(x) = xA(x)$ and the previous decompositions imply the functional equation $A(x) = x^2 + x^2 A(x) + xA(x) + xA(x)^2$, and

$$A(x) = \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x} \quad (1)$$

which generates the generalized Catalan numbers (see A004148 in [20]), which among other things, counts the peakless Motzkin paths. The first values of a_n for $2 \leq n \leq 10$ are 1, 1, 2, 4, 8, 17, 37, 82, 185.

If a Dyck path with air pockets $\alpha \in \mathcal{A}_n$ has $k \geq 1$ peaks (a peak is an occurrence UD_i for some $i \geq 1$), then it contains $n - k$ up-steps. If we ‘unfurl’

all of its down-steps D_i , $i \geq 1$, into runs D^i of i consecutive D -steps, then we obtain a Dyck path of length $2(n - k)$ having k peaks. This gives rise to a bijection between Dyck paths of semilength $n - k$ with k peaks and n -length Dyck paths with air pockets with k peaks. Hence, the number of n -length Dyck paths with air pockets with k peaks is equal to the Narayana number $N(n - k, k) = \frac{1}{n-k} \binom{n-k}{k} \binom{n-k}{k-1}$ (see [7]).

In the following, a *pattern* consists of consecutive steps in a path, and a *statistic* \mathbf{s} is an integer-valued function from a set \mathcal{S} of paths. To a given pattern p , we associate the pattern statistic $\mathbf{p} : \mathcal{S} \rightarrow \mathbb{N}$ where $\mathbf{p}(a)$ is the number of occurrences of the pattern p in $a \in \mathcal{S}$ (we use the boldface to denote statistics). For example, the statistic giving the number of occurrences of the consecutive pattern UU in a path is denoted by \mathbf{UU} . For $n \geq 1$, we denote by $\hat{\mathbf{n}}$ the constant statistic returning the value n . The *popularity* of a pattern p in \mathcal{S} is the total number of occurrences of p over all objects of \mathcal{S} , that is $\mathbf{p}(\mathcal{S}) = \sum_{a \in \mathcal{S}} \mathbf{p}(a)$ ([6, 9, 10]). Let \mathcal{S}' be a set of combinatorial objects, we say that two statistics, \mathbf{s} on \mathcal{S} and \mathbf{t} on \mathcal{S}' , have the *same distribution* if there exists a bijection $f : \mathcal{S} \rightarrow \mathcal{S}'$ satisfying $\mathbf{s}(a) = \mathbf{t}(f(a))$ for any $a \in \mathcal{S}$. In this case, with a slight abuse of the notation already used in [5], we write $f(\mathbf{s}) = \mathbf{t}$ or $\mathbf{s} = \mathbf{t}$ whenever f is the identity.

The remainder of this paper is organized as follows. In Section 2, we present a constructive bijection between n -length Dyck paths with air pockets and peakless Motzkin paths of length $n - 1$, and we show how this bijection transports some statistics. In Section 3, we provide bivariate generating functions $A(x, y) = \sum_{n, k \geq 0} a_{n, k} x^n y^k$ for the distributions of some statistics \mathbf{s} , i.e. the coefficient $a_{n, k}$ of $x^n y^k$ is the number of paths $\alpha \in \mathcal{A}_n$ satisfying $\mathbf{s}(\alpha) = k$. Then, we deduce the popularities of some patterns (U , D , peak, return, catastrophe, pyramid, ...) by calculating $\partial_y(A(x, y))|_{y=1}$, and we provide asymptotic approximations for them using classical methods (see [8, 16]). We refer to Table 1 for an overview of the results. As a byproduct, we point out a link between the popularity of pyramids and a special kind of closed smooth self overlapping curves in the plane (a subset of Fibonacci meanders defined in [13, 22]). In Section 4, we make a similar study for non-decreasing Dyck paths with air pockets.

Pattern	Pattern popularity in \mathcal{A}_n	OEIS
U	1, 2, 5, 13, 32, 80, 201, 505, 1273, 3217	A110320
D	1, 0, 2, 3, 7, 17, 40, 97, 238, 587	A051291
Peak	1, 1, 3, 7, 16, 39, 95, 233, 577, 1436	A203611
Ret	1, 1, 3, 6, 13, 29, 65, 148, 341, 793	A093128
Cat	0, 1, 1, 4, 8, 19, 44, 102, 239, 563	
Δ_k	$\underbrace{0, \dots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
$\Delta_{\geq k}$	$\underbrace{0, \dots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 30, 70, 167, 405$	A201631(= u_n)
$\Delta_{\leq k}$	$\Delta_{\leq 1}$ 1, 0, 2, 3, 7, 17, 40, 97, 238, 587	$u_n - u_{n-k}$
	$\Delta_{\leq 2}$ 1, 1, 2, 5, 10, 24, 47, 137, 335, 825, ...	
	$\Delta_{\leq 3}$ 1, 1, 3, 5, 12, 27, 64, 154, 375, 922, ...	
	\vdots	

Table 1: Pattern popularity in \mathcal{A}_n , for $2 \leq n \leq 11$.

2 Bijection with peakless Motzkin paths

In this section we exhibit a constructive bijection between n -length Dyck paths with air pockets and $(n - 1)$ -length peakless Motzkin paths, i.e. lattice paths in the first quarter plane starting at the origin, ending at $(n - 1, 0)$, made of U , D and $F = (1, 0)$ and having no occurrence of UD . Moreover, we show how our bijection transports some pattern based statistics. We denote by \mathcal{M}_n the set of peakless Motzkin paths of length n , and $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$.

Definition 1. *We recursively define the map ψ from \mathcal{A} to \mathcal{M} as follows. For $\alpha \in \mathcal{A}$, we set:*

$$\psi(\alpha) = \begin{cases} F & \text{if } \alpha = UD, & (i) \\ U\psi(\beta)D & \text{if } \alpha = \beta UD \text{ with } \beta \in \mathcal{A}, & (ii) \\ \psi(\alpha^b)F & \text{if } \alpha \in \mathcal{P}, & (iii) \\ \psi(\gamma^b)U\psi(\beta)D & \text{if } \alpha = \beta\gamma \text{ with } \beta \in \mathcal{A} \text{ and } \gamma \in \mathcal{P}. & (iv) \end{cases}$$

Due to the recursive definition, the image by ψ of a n -length Dyck path with air pockets is a peakless Motzkin path of length $n - 1$. For instance, the images of UD , UUD_2 , $UUUD_2UD_2UD$ are respectively F , FF , and $UUFFDFD$. We refer to Figure 1 for an illustration of this mapping.

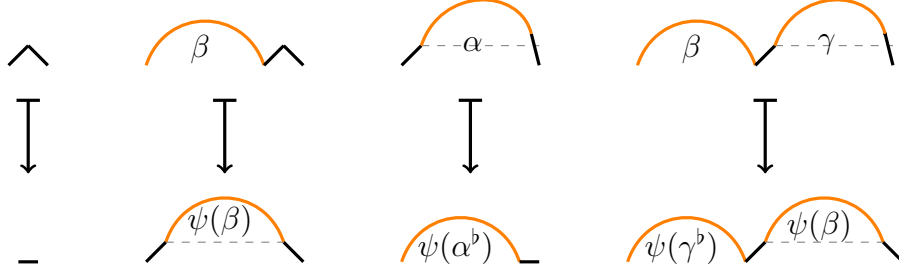


Figure 1: Illustration of the map ψ according to Definition 1.

Theorem 1. *For all $n \geq 2$, the map ψ induces a bijection between \mathcal{A}_n and \mathcal{M}_{n-1} .*

Proof. It is well known that the cardinality of \mathcal{M}_n is given by the n -th term of generalized Catalan number (see A004148 in [20]). So it suffices (see observation after relation (1)) to prove the injectivity of ψ . We proceed by induction on n . The case $n = 2$ is obvious since $\mathcal{A}_2 = \{UD\}$ and $\mathcal{M}_1 = \{F\}$. For all $k \leq n$, we assume that ψ is an injection from \mathcal{A}_k to \mathcal{M}_{k-1} , and we prove the result for $k = n + 1$. According to Definition 1, if α and β in \mathcal{A}_{n+1} satisfy $\psi(\alpha) = \psi(\beta)$, then α and β necessarily come from the same case among (i) – (iv). Using the induction hypothesis, we conclude directly that $\alpha = \beta$, which completes the induction. Thus ψ is injective and so bijective. \square

Proposition 1. *For all $n \geq 2$ and $k \geq 1$, and $\psi : \mathcal{A}_n \rightarrow \mathcal{M}_{n-1}$, the following holds:*

- $\psi(\mathbf{U}) = \mathbf{F} + \mathbf{U} = \mathbf{F} + \mathbf{D}$
- $\psi(\mathbf{D}) = \psi(\mathbf{UD}) = \mathbb{1}_{\mathbf{F}} + \mathbf{UFD} + \mathbb{1}_{\mathbf{U,MD}} + \mathbf{U}^2\mathbf{MD}^2$
- $\psi(\mathbf{DU}) = \mathbf{UFD} + \mathbf{U}^2\mathbf{MD}^2$
- $\psi(\mathbf{UU}) = \mathbf{F} - \hat{\mathbf{1}}$
- $\psi(\mathbf{\Delta}_k) = \mathbb{1}_{\mathbf{F}^k} + \mathbf{UF}^k\mathbf{D} + \mathbb{1}_{\mathbf{F}^{k-1}\mathbf{U,MD}} + \mathbf{UF}^{k-1}\mathbf{U,MD}^2$
- $\psi(\mathbf{Peak}) = \mathbf{U} + \hat{\mathbf{1}}$
- $\psi(\mathbf{Ret}) = \hat{\mathbf{n}} - \mathbf{LastF}$
- $\psi(\mathbf{SLast}) = \mathbf{Ret}$,

where $\mathbb{1}_\beta(\alpha) = 1$ if $\alpha = \beta$ and 0 otherwise; $\mathbb{1}_{\mathbf{U}\mathcal{M}\mathbf{D}}(\alpha)$ is equal to 1 if there exists $\beta \in \mathcal{M}$ such that $\alpha = U\beta D$ and 0 otherwise; $\mathbf{U}^2\mathcal{M}\mathbf{D}^2(\alpha)$ is the number of occurrences $U^2\beta D^2$ in α for $\beta \in \mathcal{M}$; $\Delta_{\mathbf{k}}(\alpha)$ is the number of occurrences $U^{\mathbf{k}}D_{\mathbf{k}}$ in α ; $\mathbf{Peak}(\alpha) = \sum_{\mathbf{k} \geq 1} \mathbf{U}\mathbf{D}_{\mathbf{k}}(\alpha)$; $\mathbf{Ret}(\alpha)$ is the number of returns to the x -axis of α ; $\mathbf{LastF}(\alpha)$ is the position of the rightmost flat-step in α , and $\mathbf{SLast}(\alpha)$ is the size of the the last step of α (i.e. k if the last step is D_k).

Proof. We provide the proof for $\psi(\mathbf{U})$ and $\psi(\Delta_{\mathbf{k}})$ since the those for the other relations can be obtained *mutatis mutandis*.

We proceed by induction on n . Since $\mathcal{A}_2 = \{UD\}$ and $\mathcal{M}_1 = \{F\}$ the statements trivially hold for $n = 2$. Now, assume the statements are true for all $k \leq n$ and let us prove them for $n + 1$.

If $\alpha \in \mathcal{A}_{n+1}$ with $n \geq 2$, then we have either (i) $\alpha = \beta UD$, (ii) $\alpha = \gamma^\#$ or (iii) $\alpha = \beta\gamma^\#$ where $\beta, \gamma \in \mathcal{A}$. In the case (i), $\mathbf{U}(\beta UD) = 1 + \mathbf{U}(\beta)$ and with the induction hypothesis, $\mathbf{U}(\beta UD) = 1 + (\mathbf{U} + \mathbf{F})(\psi(\beta)) = (\mathbf{U} + \mathbf{F})(U\psi(\beta)D) = (\mathbf{U} + \mathbf{F})(\psi(\beta UD))$ as expected. In the case (ii), $\mathbf{U}(\gamma^\#) = 1 + \mathbf{U}(\gamma)$ and with the induction hypothesis, $\mathbf{U}(\gamma^\#) = 1 + (\mathbf{U} + \mathbf{F})(\psi(\gamma)) = (\mathbf{U} + \mathbf{F})(\psi(\gamma)F) = (\mathbf{U} + \mathbf{F})(\psi(\gamma^\#))$. Case (iii) is handled in the same way.

So, we have $\psi(\mathbf{U}) = \mathbf{U} + \mathbf{F}$. Using a similar reasoning, we can easily prove $\psi(\mathbf{D}) = \mathbb{1}_{\mathbf{F}} + \mathbf{U}\mathbf{F}\mathbf{D} + \mathbb{1}_{\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U}^2\mathcal{M}\mathbf{D}^2$.

Now, let us give details for the slightly less straightforward case of $\psi(\Delta_{\mathbf{k}})$ for $k \geq 1$. The case $k = 1$ is already handled since we have $\psi(\mathbf{UD}) = \psi(\mathbf{D})$. So, we assume $k \geq 2$. We consider the following case analysis: any given Dyck path with air pockets is either of the form (i) βUD , (ii) $\beta\Delta_{k-1}^\#$, (iii) $\beta\Delta_k^\#$, (iv) $\beta(\alpha\Delta_k)^\#$ with $\alpha \in \mathcal{A}$, or (v) $\beta\alpha^\#$ with $\alpha \in \mathcal{A}$ being neither Δ_{k-1} , nor Δ_k , nor $\alpha'\Delta_k$ ($\alpha' \in \mathcal{A}$), and $\beta \in \mathcal{A} \cup \{\varepsilon\}$. Reasoning by induction, case (ii) unfolds as follows: if $\beta = \varepsilon$, then we get

$$(\mathbb{1}_{\mathbf{F}^k} + \mathbf{U}\mathbf{F}^k\mathbf{D} + \mathbb{1}_{\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U}\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}^2)(F^k) = 1,$$

which is the same as $\Delta_{\mathbf{k}}(\Delta_{k-1}^\#)$. Otherwise, we have

$$\begin{aligned} & (\mathbb{1}_{\mathbf{F}^k} + \mathbf{U}\mathbf{F}^k\mathbf{D} + \mathbb{1}_{\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U}\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}^2)(\psi(\beta\Delta_{k-1}^\#)) = \\ & = (\mathbb{1}_{\mathbf{F}^k} + \mathbf{U}\mathbf{F}^k\mathbf{D} + \mathbb{1}_{\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}} + \mathbf{U}\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}^2)(F^{k-1}U\psi(\beta)D) = \\ & = \mathbb{1}_{\mathbf{F}^k}(\psi(\beta)) + \mathbf{U}\mathbf{F}^k\mathbf{D}(\psi(\beta)) + 1 + \mathbb{1}_{\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}}(\psi(\beta)) + \\ & \quad + \mathbf{U}\mathbf{F}^{k-1}\mathbf{U}\mathcal{M}\mathbf{D}^2(\psi(\beta)) = \\ & = 1 + \Delta_{\mathbf{k}}(\beta) = \Delta_{\mathbf{k}}(\beta\Delta_{k-1}^\#). \end{aligned}$$

The four remaining cases are obtained in the same way. \square

3 Distribution and popularity of patterns

3.1 The numbers of U and D

Theorem 2. Let $A(x, y, z) = \sum_{n,k,\ell \geq 0} a_{n,k,\ell} x^n y^k z^\ell$ be the generating function (g.f.) where $a_{n,k,\ell}$ is the number of paths in \mathcal{A}_n having k up-steps U and ℓ down-steps $D = D_1$. Then the following holds:

$$A(x, y, z) = \frac{1 - xy - x^2yz - 2x^3y^2 + 2x^3y^2z - \sqrt{R}}{2xy(1 + x^2y - x^2yz)},$$

with

$$R = x^4y^2z^2 + 2x^3y^2z - 4x^3y^2 + x^2y^2 - 2x^2yz - 2xy + 1.$$

Proof. Any Dyck path with air pockets has one of the following forms: (i) $UD\gamma$, (ii) $U^2D_2\gamma$, (iii) $(\alpha UD)^\# \gamma$ with $\alpha \in \mathcal{A}$, (iv) $\alpha^\# \gamma$ with $\alpha \in \mathcal{A}$ not being UD nor βUD ($\beta \in \mathcal{A}$), where $\gamma \in \mathcal{A} \cup \{\varepsilon\}$. These four cases are disjoint and cover \mathcal{A} entirely. Then, we deduce the functional equation by taking into account the length, and the numbers of U and D with respect to x , y and z :

$$A = \left(\underbrace{x^2yz}_{(i)} + \underbrace{x^3y^2}_{(ii)} + \underbrace{x^3y^2A}_{(iii)} + \underbrace{xy(A - x^2yz(1 + A))}_{(iv)} \right) (1 + A),$$

where A stands for $A(x, y, z)$. Solving for A , we get the result. \square

Corollary 1. For all $n \geq 1$, the number of Dyck paths with air pockets (of any length) having n up-steps U is the n -th Catalan number $\frac{1}{n+1} \binom{2n}{n}$ (see A000108 in [20]).

Proof. We check that $1 + A(1, y, 1)$ is the g.f. of the Catalan numbers. \square

Corollary 2. For all $n \geq 1$, the number of Dyck paths with air pockets having n up-steps U and no down-steps D is the n -th Riordan number $\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} c_k$, where $c_k = \frac{1}{k+1} \binom{2k}{k}$ (see A005043 in [20]).

Proof. We check that $1 + A(1, y, 0)$ is the g.f. of the Riordan numbers. \square

Corollary 3. The g.f. for the popularity of up-steps U in \mathcal{A}_n is

$$\frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x\sqrt{x^4 - 2x^3 - x^2 - 2x + 1}},$$

which generates a shift of the sequence A110320 in [20]. An asymptotic approximation of the n -th term is

$$\frac{2(\sqrt{5}-2)}{(3-\sqrt{5})\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{3+\sqrt{5}}{2}\right)^n.$$

Proof. The g.f. is given by $\partial_y(A(x, y, 1))|_{y=1}$. The asymptotic approximation is obtained using classical methods (see [8, 16]) \square

Corollary 4. *The g.f. for the popularity of down-steps $D = D_1$ in \mathcal{A}_n is*

$$\frac{x^2(1+2x^2-x^3+(1-x)\sqrt{x^4-2x^3-x^2-2x+1})}{2\sqrt{x^4-2x^3-x^2-2x+1}},$$

which generates a shift of the sequence A051291 in [20]. An asymptotic approximation of the n -th term is

$$\frac{(\sqrt{5}-3)^2(\sqrt{5}-1)}{8\sqrt{\pi n}\sqrt{14\sqrt{5}-30}}\left(\frac{3+\sqrt{5}}{2}\right)^n.$$

Proof. The g.f. is given by $\partial_z(A(x, 1, z))|_{z=1}$. \square

3.2 The number of peaks

In this part, we study the distribution of peaks, i.e. patterns UD_m for $m \geq 1$.

Theorem 3. *Let $P(x, y) = \sum_{n,k \geq 0} p_{n,k} x^n y^k$ be the g.f. where $p_{n,k}$ is the number of n -length Dyck paths with air pockets having k peaks. Then we have:*

$$P(x, y) = \frac{1-x-x^2y-\sqrt{(1-x-x^2y)^2-4x^3y}}{2x},$$

which generates a shift of the sequence A089732 in [20].

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $P(x, y)$ is $(1+P(x, y))x^2y$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1+P(x, y))xP(x, y)$. Hence, the second-to-last return decomposition yields:

$$P(x, y) = (1+P(x, y))(x^2y + xP(x, y)),$$

which gives the result after solving for $P(x, y)$. \square

Corollary 5. *The g.f. for the popularity of peaks in \mathcal{A}_n is*

$$\frac{x \left(1 + x - x^2 - \sqrt{(1 - x - x^2)^2 - 4x^3} \right)}{2\sqrt{(1 - x - x^2)^2 - 4x^3}},$$

which generates a shift of the sequence A203611 in [20]. An asymptotic approximation of the n -th term is

$$\frac{(3 - \sqrt{5})(\sqrt{5} - 1)}{4\sqrt{\pi n} \sqrt{14\sqrt{5} - 30}} \left(\frac{3 + \sqrt{5}}{2} \right)^n.$$

Remark 1. *Another way of finding the total number of peaks in all Dyck paths with air pockets of length n is the following: since the number of n -length Dyck paths with air pockets with k peaks is $N(n - k, k)$, we have:*

$$\mathbf{Peak}(\mathcal{A}_n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} k N(n - k, k) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n - k} \binom{n - k}{k} \binom{n - k}{k - 1}.$$

Using the formula for the sequence A203611 in [20], we get the following identity:

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n - k - 1}{k - 1} \binom{n - k}{k - 1} = \sum_{k=0}^{n-1} \binom{k - 1}{2k - n} \binom{k}{2k - n + 1}.$$

3.3 The number of returns to the x -axis

A return to the x -axis is a step D_m , $m \geq 1$, ending on the x -axis.

Theorem 4. *Let $R(x, y) = \sum_{n, k \geq 0} r_{n, k} x^n y^k$ be the g.f. where $r_{n, k}$ is the number of n -length Dyck paths with air pockets with k returns, then:*

$$R(x, y) = \frac{2}{2 - y \left(1 - x + x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1} \right)} - 1,$$

which generates the triangle A098086 in [20] where the row n and column k gives the number of peakless Motzkin paths having its leftmost F -step on the k -th step (see also Proposition 1).

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $R(x, y)$ is $(1 + R(x, y))x^2y$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1 + R(x, y))xyA(x)$. So we deduce,

$$R(x, y) = (1 + R(x, y))(x^2y + xyA(x)),$$

which gives the result using relation (1). \square

Corollary 6. *The g.f. for the popularity of returns to the x -axis in \mathcal{A}_n is*

$$2 \frac{1 - x + x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{(1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})^2},$$

which corresponds to the sequence A093128 in [20], where the n -th term counts all possible dissections of a regular $(n + 2)$ -gon using zero or more strictly disjoint diagonals. An asymptotic approximation of the n -th term is

$$\frac{\sqrt{14\sqrt{5} - 30}\sqrt{5}}{4n\sqrt{\pi n}} \left(\frac{3 + \sqrt{5}}{2} \right)^{n+1}.$$

3.4 The number of catastrophes

A catastrophe is a step D_m , $m \geq 2$, ending on the x -axis.

Theorem 5. *Let $C(x, y) = \sum_{n,k \geq 0} c_{n,k} x^n y^k$ be the g.f. where $c_{n,k}$ is the number of n -length Dyck paths with air pockets with k catastrophes. Then we have:*

$$C(x, y) = \frac{2}{2 - 2x^2 - y(1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})} - 1.$$

The first few terms of its series expansion are: $x^2 + x^3y + x^4(y + 1) + 4x^5y + x^6(y^2 + 6y + 1) + x^7y(2y + 15) + O(x^8)$ which does not appear in [20].

Proof. If a Dyck path with air pockets equals αUD with $\alpha \in \mathcal{A} \cup \{\varepsilon\}$, then its contribution to $C(x, y)$ is $(1 + C(x, y))x^2$; if it has the form $\alpha\beta^\sharp$ with $\beta \in \mathcal{A}$, then its contribution is $(1 + C(x, y))xyA(x)$. So, we deduce $C(x, y) = (1 + C(x, y))(x^2 + xyA(x))$. \square

Corollary 7. *The g.f. for the popularity of catastrophes in \mathcal{A}_n equals*

$$2 \frac{1 - x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{(1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1})^2}.$$

The first few terms of its series expansion are: $x^3 + x^4 + 4x^5 + 8x^6 + 19x^7 + 44x^8 + O(x^9)$, and an asymptotic approximation of the n -th term is

$$\frac{\sqrt{14\sqrt{5}-30}(4-\sqrt{5})}{4n\sqrt{\pi n}} \left(\frac{3+\sqrt{5}}{2}\right)^{n+1}.$$

Remark 2. As a byproduct of Corollaries 6 and 7, the ratio of the popularity of catastrophes in \mathcal{A}_n to the popularity of returns in \mathcal{A}_n tends to $\frac{4-\sqrt{5}}{\sqrt{5}} = 0.788854\dots$ when n tends toward ∞ .

3.5 The number of pyramids $U^k D_k$

A k -pyramid Δ_k in a path is an occurrence of the pattern $U^k D_k$, $k \geq 1$.

Theorem 6. For all $k \geq 1$, the g.f. $P_k(x, y) = \sum_{n,m \geq 0} p_{n,m}^k x^n y^m$ where $p_{n,m}^k$ is the number of n -length Dyck paths with air pockets having m k -pyramids is given by:

$$P_k(x, y) = \frac{x^{k+1}(y-1) - 2x^{k+2}(y-1) + x^2 + x - 1 + \sqrt{Q}}{2(x^{k+2}(y-1) - x)},$$

where

$$Q = x^{k+1}(y-1)(x^{k+1}(y-1) + 4x + 2(x^2 - x - 1)) + (x^2 + x + 1)(x^2 - 3x + 1).$$

Proof. We refine the second-to-last return decomposition so that any Dyck path with air pockets falls into one of the following cases: (i) $\Delta_m \gamma$ with $1 \leq m \leq k-1$, (ii) $\Delta_k \gamma$, (iii) $\Delta_{k+1} \gamma$, (iv) $(\alpha \Delta_k)^\# \gamma$ with $\alpha \in \mathcal{A}$, (v) $\beta^\# \gamma$ with $\beta \in \mathcal{A}$ not being Δ_m with $1 \leq m \leq k$, nor $(\alpha \Delta_k)^\#$ with $\alpha \in \mathcal{A}$, where $\gamma \in \mathcal{A} \cup \{\varepsilon\}$. These five cases are disjoint and cover all Dyck paths with air pockets. So, we deduce:

$$P_k = \left(\underbrace{\sum_{i=2}^k x^i}_{(i)} + \underbrace{x^{k+1}y}_{(ii)} + \underbrace{x^{k+2}}_{(iii)} + \underbrace{x^{k+2}P_k}_{(iv)} + x \cdot \underbrace{\left(P_k - \sum_{i=2}^k x^i - x^{k+1}y(1+P_k) \right)}_{(v)} \right) (1 + P_k),$$

where P_k stands for $P_k(x, y)$. Solving for P_k , we get the result. \square

Corollary 8. For $k \geq 1$, the g.f. for the popularity $\Delta_{\mathbf{k}}(\mathcal{A}_n)$ of k -pyramids in \mathcal{A}_n equals:

$$Y_k(x) = \frac{x^{k+1} \left(1 + 2x^2 - x^3 + (1-x)\sqrt{(x^2+x+1)(x^2-3x+1)} \right)}{2\sqrt{(x^2+x+1)(x^2-3x+1)}},$$

which generates the $(n-k-2)$ -th term of the sequence A051291 in [20]. In particular, we have $\Delta_{\mathbf{1}}(\mathcal{A}_n) = \Delta_{\mathbf{k}}(\mathcal{A}_{n+k-1})$ for all $k \geq 1$ and $n \geq 2$, which means that there are as many 1-pyramids in \mathcal{A}_n as there are k -pyramids in \mathcal{A}_{n+k-1} . An asymptotic approximation of the n -th term of this sequence is

$$\frac{\sqrt{5}-1}{2\sqrt{\pi n}\sqrt{14\sqrt{5}-30}} \left(\frac{3+\sqrt{5}}{2} \right)^{n-k-1}.$$


An immediate consequence of Corollary 8 is the following.

Corollary 9. For $k \geq 1$, the g.f. for the popularities $\Delta_{\geq \mathbf{k}}(\mathcal{A}_n)$ and $\Delta_{\leq \mathbf{k}}(\mathcal{A}_n)$ are respectively given by

$$Y_{\geq k}(x) = \frac{x^{k-1}}{1-x} Y_1(x) \quad \text{and} \quad Y_{\leq k}(x) = \frac{1-x^k}{1-x} Y_1(x),$$

which means that $\Delta_{\leq \mathbf{k}}(\mathcal{A}_{n-k+1}) = \Delta_{\geq \mathbf{k}}(\mathcal{A}_n) - \Delta_{\geq \mathbf{k}}(\mathcal{A}_{n-k})$.

For any $k \geq 1$, the popularity of pyramids of size at least k in \mathcal{A}_n (see Corollary 9) seems to correspond to a shift of the sequence A203611 in [20], which enumerates *Fibonacci meanders with central angle 180 degrees* (see Luschny's [13] and Wienand's [22] posts in OEIS Wiki about meanders). In order to prove this fact, we give the formal definition of such meanders, and provide their g.f. that does not exist in the literature (to our knowledge).

A Fibonacci meander with central angle 180 degrees is a closed smooth self-overlapping curve in the plane, consisting of an even length sequence of two types of arcs of angle 180 degrees, namely $L = \smile$ and $R = \frown$, starting at the origin with an L -arc toward the north, having no consecutive L -arcs except at the beginning where a run (of any length) of consecutive L -arcs is authorized. Each arc starts at the end of the previous arc and it preserves the direction of its arrow, i.e. LLR corresponds to . Let \mathcal{F}_{2n} be the set of such meanders of length $2n$. For instance, the left part of Figure 2 illustrates a meander in \mathcal{F}_{20} .

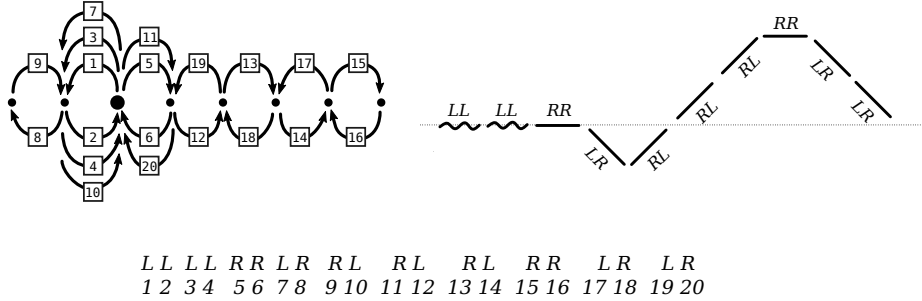


Figure 2: A Fibonacci meander in \mathcal{F}_{20} where the sequence of arcs is given by $LLLLRRLRRLRLRLRRLRLR$, and its associated lattice path.

Now, we define a function τ , mapping a two-letter word over the alphabet $\{L, R\}$ into the set $\{U, D, F, \tilde{F}\}$:

$$\tau(a) = \begin{cases} U, & \text{if } a = RL, \\ D, & \text{if } a = LR, \\ F, & \text{if } a = RR, \\ \tilde{F}, & \text{if } a = LL, \end{cases}$$

and a function μ , mapping a meander $w = w_1 w_2 \dots w_{2n} \in \mathcal{F}_{2n}$ into an n -length word over the alphabet $\{U, D, F, \tilde{F}\}$:

$$\mu(w) = \tau(w_1 w_2) \tau(w_3 w_4) \dots \tau(w_{2n-1} w_{2n}).$$

Grand Motzkin paths are lattice paths of length n in $\mathbb{N} \times \mathbb{Z}$, consisting of steps $U = (1, 1)$, $D = (1, -1)$, and $F = (1, 0)$, starting at $(0, 0)$, ending at $(n, 0)$ (contrarily to classical Motzkin paths, they can go below the x -axis). For instance $DUUDDFDUUUD$ is a grand Motzkin path of length 11. Let \mathcal{G}_n be the set of peakless (i.e. with no occurrence of UD) grand Motzkin paths of length n and \mathcal{N}_n be the subset of paths starting with a D -step in \mathcal{G}_n . Denote by \mathcal{S}_n the set of peakless grand Motzkin paths of length n starting with a down step or prefixed by a non empty sequence of special flats, called *wavy flats*, $\tilde{F} = (1, 0)$. For instance, $\tilde{F}\tilde{F}DUUDDFDUUUD \in \mathcal{S}_{13}$.

Proposition 2. *The function μ induces a bijection from \mathcal{F}_{2n} to \mathcal{S}_n .*

Proof. Every meander $a \in \mathcal{F}_{2n}$ avoids the pattern LL except if the occurrence of LL is only preceded by letters L , which means that $\mu(a)$ avoids the pattern UD . If the meander a starts with a maximal prefix of the form $(LL)^k$, $k \geq 1$, then $\mu(a)$ starts, with a maximal sequence of k wavy flats, \tilde{F}^k .

If the meander a starts with LR , then $\mu(a)$ starts with D . Moreover, the image by μ of a factor RR is $F = \tau(RR)$ in $\mu(a)$, the image of LR is D and the image of RL is U . Thus, the fact that a is a closed curve, implies that $\mu(a)$ starts and ends on x -axis. Due to all these observations, μ is necessarily a bijection from \mathcal{F}_{2n} to \mathcal{S}_n . \square

Theorem 7. *The g.f. $S(x) = \sum_{n \geq 0} s_n x^n$, where the coefficient s_n is the number of $2n$ -length Fibonacci meanders with a central angle 180 degrees, is*

$$S(x) = \frac{x^2 - x + 1 - \sqrt{R}}{(x-1)(x^2+x+1)(x^2-3x+1) + (x-1)(x^2-x-1)\sqrt{R}},$$

with $R = (x^2 + x + 1)(x^2 - 3x + 1)$. Using Corollary 9, we have

$$S(x) = \frac{Y_1(x)}{x^2(1-x)} - 1,$$

which establishes the expected link between Fibonacci meanders and the popularity $\Delta_{\geq \mathbf{k}}(\mathcal{A}_n)$.

Proof. Considering Proposition 2, it suffices to enumerate \mathcal{S}_n . We set $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}_n$, $\mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}_n$, and $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$. Recall that \mathcal{M} is the set of non-empty peakless Motzkin paths. Denote by \mathcal{V} the set of Motzkin paths without valleys DU and by $\bar{\mathcal{V}}$ the set of paths obtained by symmetry about the x -axis ($U \leftrightarrow D$) of valleyless Motzkin paths, e.g. $DUFDU \in \bar{\mathcal{V}}$ since it is symmetry of $UDFUD \in \mathcal{V}$. Let \mathcal{W} be the set of non empty sequences of wavy flat steps. We use $M(x)$, $V(x) = \bar{V}(x)$, $G(x)$, $N(x)$, $W(x)$ to denote the corresponding generating functions with respect to the length.

Obviously, we have $W(x) = \frac{x}{1-x}$. From relation (1) and Theorem 1 we obtain $M(x) = A(x)/x$. Due to the fact that a nonempty path $P \in \mathcal{V}$ is either of the form FQ , or UQD , or $UQDFR$, with $Q, R \in \mathcal{V}$, we deduce $V(x) = 1 + xV(x) + x^2V(x) + x^3V(x)^2$. Finally, we decompose \mathcal{G} , \mathcal{N} and \mathcal{S} as illustrated below:

$$\begin{aligned}
\mathcal{G} &= \varepsilon \uplus - \circlearrowleft \mathcal{G} \uplus \overset{\mathcal{M}}{\curvearrowright} \circlearrowleft \mathcal{G} \uplus \circlearrowleft \mathcal{N}, \\
\mathcal{N} &= \curvearrowleft \bar{\mathcal{V}} \uplus \curvearrowleft \bar{\mathcal{V}} \uplus \overset{\mathcal{M}}{\curvearrowright} \circlearrowleft \mathcal{G} \uplus \curvearrowleft \bar{\mathcal{V}} \uplus \circlearrowleft \mathcal{G}, \\
\mathcal{S} &= \overset{\mathcal{W}}{\curvearrowright} \circlearrowleft \mathcal{G} \uplus \circlearrowleft \mathcal{N},
\end{aligned}$$

which induces the following system of functional equations

$$\begin{cases}
G(x) &= 1 + xG(x) + x^2M(x)G(x) + N(x), \\
N(x) &= x^2V(x) + x^4V(x)M(x)G(x) + x^3V(x)G(x), \\
S(x) &= W(x)G(x) + N(x).
\end{cases}$$

Solving this system, we obtain $S(x)$. □

4 Non-decreasing Dyck paths with air pockets

A Dyck path with air pockets is *non-decreasing* if the sequence of heights of its valleys is non-decreasing, i.e. the sequence of the minimal ordinates of the occurrences D_kU , $k \geq 1$, is non-decreasing from left to right. See [2] for a reference about non-decreasing Dyck paths. For example, the Dyck path with air pockets $UUDUDUD_2$ is non-decreasing, since its two valleys both lie at height 1, while the path $UUDUD_2UD$ is not, since its two valleys lie at heights 1 and 0 from left to right. Let \mathcal{A}_n^\nearrow , $n \geq 2$, be the set of n -length non-decreasing Dyck paths with air pockets and $\mathcal{A}^\nearrow = \bigcup_{n \geq 2} \mathcal{A}_n^\nearrow$. The subset of n -length prime non-decreasing Dyck paths with air pockets is defined as the intersection $\mathcal{P}_n^\nearrow := \mathcal{A}_n^\nearrow \cap \mathcal{P}$, and we set $\mathcal{P}^\nearrow := \bigcup_{n \geq 2} \mathcal{P}_n^\nearrow$. Analogous to generic Dyck paths with air pockets, the map $\alpha \mapsto \alpha^\flat$ induces a bijection between \mathcal{P}_n^\nearrow and $\mathcal{A}_{n-1}^\nearrow$, whose inverse is the map $\alpha \mapsto \alpha^\sharp$.

Theorem 8. *For $n \geq 2$, if a_n^\nearrow is the number of n -length non-decreasing Dyck paths with air pockets, then $a_2^\nearrow = 1$ and $a_n^\nearrow = 2^{n-3}$ for $n \geq 3$.*

Proof. Any non-decreasing Dyck path with air pockets α has one of the following three forms: (i) $\alpha = UD\beta$ with $\beta \in \mathcal{A}^\nearrow$, (ii) $\alpha \in \mathcal{P}^\nearrow \cup \{UD\}$, or (iii) $\alpha = \Delta_k\beta$ where $k \geq 2$ and $\beta \in \mathcal{A}^\nearrow$. So, if $A^\nearrow(x) = \sum_{n \geq 2} a_n^\nearrow x^n$ where a_n^\nearrow is the cardinality of \mathcal{A}_n^\nearrow , then the previous decomposition implies the functional equation

$$A^\nearrow(x) = x^2 A^\nearrow(x) + x(A^\nearrow(x) + x) + \frac{x^3}{1-x} A^\nearrow(x).$$

Thus we have $A^\nearrow(x) = \frac{x^2(1-x)}{1-2x}$ which completes the proof. \square

Theorem 9 gives the enumeration of a particular subclass of \mathcal{A}^\nearrow where all valleys are at height zero.

Theorem 9. *For all $n \geq 1$, the number of n -length non-decreasing Dyck paths with air pockets which have no valley with positive height is equal to F_{n-1} , where F_k is the k -th Fibonacci number.*

Proof. Any Dyck path with air pockets having no valley at positive height, has the form $U^k D_k \alpha$, $k \geq 1$, where α is either empty or a path having no valley at positive height. If $A_0(x)$ is the g.f. of such paths, then we deduce $A_0(x) = \frac{x^2}{1-x}(1 + A_0(x))$ which implies that $A_0(x) = \frac{x^2}{1-x-x^2}$. \square

4.1 The numbers of U and D

Theorem 10. *Let $A^\nearrow(x, y, z) = \sum_{n, k, \ell \geq 0} a_{n, k, \ell}^\nearrow x^n y^k z^\ell$ be the trivariate g.f. where $a_{n, k, \ell}^\nearrow$ is the number of n -length non-decreasing Dyck paths with air pockets having k up-steps U and ℓ down-steps D . Then, $A^\nearrow(x, y, z)$ equals*

$$\frac{x^2 y (1 - xy) (xyz - xy - z) (x^2 yz + xy - 1)}{(x^3 y^2 (z - 1) + x^2 y (y - z) - 2xy + 1) (x^3 y^2 (z - 1) - x^2 yz - xy + 1)}.$$

The first few terms of the series expansion are $x^2 yz + x^3 y^2 + x^4 (y^4 + y^2 z^2) + x^5 (y^4 + 3y^3 z) + x^6 (y^5 + 4y^4 z + 2y^4 + y^3 z^3) + O(x^7)$.

Proof. Let $Z(x, y, z) = \sum_{n, k, \ell \geq 0} z_{n, k, \ell} x^n y^k z^\ell$, where $z_{n, k, \ell}$ is the number of n -length non-decreasing Dyck paths with air pockets having only valleys at height 0, k up-steps U and ℓ down-steps D . Such a path has the form $UD\alpha$ or $U^k D_k \alpha$ with $k \geq 2$, where α has all its valleys at height 0. Then, we have

$$Z(x, y, z) = (1 + Z(x, y, z)) \left(x^2 yz + \frac{x^3 y^2}{1 - xy} \right).$$

Solving for $Z(x, y, z)$, we get:

$$Z(x, y, z) = \frac{x^2y(xy(1-z) + z)}{x^3y^2(z-1) - x^2yz - xy + 1}.$$

Now, any non-decreasing Dyck path with air pockets belongs to one of the following cases: (i) βUD , (ii) $\beta(UD)^\sharp$, (iii) $\beta(\alpha UD)^\sharp$ (α having all its valleys at height 0), (iv) $\beta\alpha^\sharp$ (α having all its valleys at height 0, and not ending with UD), where β is either empty or has all its valleys at height 0. Thus, we have (for short, we use A^∇ and Z instead of $A^\nabla(x, y, z)$ and $Z(x, y, z)$):

$$A^\nabla = (1 + Z)(x^2yz + x^3y^2 + x^3y^2Z + xy(A^\nabla - x^2yz(1 + Z))).$$

Solving for A^∇ , we get the result. \square

As we have made in Section 3.1, we deduce the following.

Corollary 10. *For all $n \geq 1$, the number of non-decreasing Dyck paths with air pockets:*

- *having n up-steps U is the n -th term of the sequence A001519;*
- *having n up-steps U and no down-steps D is the $(n-1)$ -th term of the sequence A099036.*

Corollary 11. *The popularity of up-steps U in \mathcal{A}_n^∇ is equal to the $(n-2)$ -th term of the sequence A098156 in [20].*

Corollary 12. *The g.f. for the popularity of down-steps D in \mathcal{A}_n^∇ equals:*

$$\frac{x^2(1-x)(1-4x+5x^2-2x^3+x^5)}{(1-2x)^2(1-x-x^2)}.$$

An asymptotic approximation of the n -th term is $n \cdot 2^{n-6}$. The first few terms of the series expansion are $x^2 + 2x^4 + 3x^5 + 7x^6 + 15x^7 + 33x^8 + 72x^9 + O(x^{10})$.

4.2 The number of peaks

Theorem 11. *For all $n \geq 2$ and $k \geq 1$, the number of n -length non-decreasing Dyck paths with air pockets having k peaks is equal to $\binom{n-2}{2(k-1)}$.*

Proof. Let $B(x, y)$ be the g.f. where the coefficient of $x^n y^k$ is the number of n -length paths in \mathcal{A}^∇ having k peaks. Any non-decreasing Dyck path with air pockets is either of the form UD , or $UD\beta$, or α^\sharp , or $\Delta_k\beta$ with $k \geq 2$, with $\alpha, \beta \in \mathcal{A}^\nabla$. This yields the following functional equation:

$$B(x, y) = x^2y + xB(x, y) + \frac{x^2}{1-x}yB(x, y)$$

with the solution $B(x, y) = \frac{(1-x)x^2y}{(1-x)^2-x^2y}$, which generates the sequence A034839 in [20]. \square

Corollary 13. *The popularity of peaks in \mathcal{A}_n^\nearrow is the $(n-2)$ -th term of the sequence A045891 in [20], which is equal to $(n+2) \cdot 2^{n-5}$ for $n \geq 4$.*

4.3 The number of returns to the x -axis

Theorem 12. *The bivariate g.f. where the coefficient of $x^n y^k$ is the number of n -length non-decreasing Dyck paths with air pockets having k returns is*

$$R(x, y) = \frac{x^2 y (1-x)(1-x-x^2)}{(1-2x)(1-x-x^2 y)}.$$

The first few terms of the series expansion are $x^2 y + x^3 y + x^4(y^2 + y) + x^5(2y^2 + 2y) + x^6(y^3 + 3y^2 + 4y) + x^7(3y^3 + 5y^2 + 8y) + O(x^8)$.

Proof. Using the second-to-last return decomposition of \mathcal{A}^\nearrow , we easily get the following functional equation:

$$R(x, y) = x^2 y + x^2 y R(x, y) + x y A^\nearrow(x) + \frac{x^3}{1-x} y R(x, y),$$

which gives the result. \square

Corollary 14. *The g.f. for the popularity of returns in \mathcal{A}_n^\nearrow is*

$$\frac{x^2(1-x)^2}{(1-2x)(1-x-x^2)},$$

and for $n \geq 2$ the coefficient of x^n is $2^{n-2} - F_{n-2}$, where F_n is the n -th Fibonacci number (see A099036 in [20]).

4.4 The number of catastrophes

Theorem 13. *The bivariate g.f. where the coefficient of $x^n y^k$ is the number of n -length non-decreasing Dyck paths with air pockets having k catastrophes is*

$$C(x, y) = \frac{x^2(1-x)(1+x(y-2)-x^2y)}{(1-2x)(1-x-x^2-x^3(y-1))}.$$

The first few terms of the series expansion are $x^2 + x^3 y + x^4(y+1) + 4x^5 y + x^6(y^2 + 6y + 1) + x^7(2y^2 + 14y) + x^8(7y^2 + 24y + 1) + O(x^9)$.

Proof. First, let us determine the bivariate g.f. $U(x, y)$ with respect to the length and number of catastrophes for non-decreasing Dyck paths with air pockets having all their valleys at height 0. It is easy to see that

$$U(x, y) = (1 + U(x, y)) \left(x^2 + \frac{x^3 y}{1 - x} \right),$$

which yields:

$$U(x, y) = \frac{x^2(1 - x + xy)}{1 - x - x^2 - x^3(y - 1)}.$$

Then, any non-decreasing Dyck path with air pockets has one of the following forms: (i) βUD , or (ii) $\beta U^2 D_2$, or (iii) $\beta \alpha^\#$ with α not belonging to forms (i) or (ii), and where β is either empty or a non-decreasing Dyck path with air pockets which only has valleys that lie at height 0. Hence, the bivariate generating function $C(x, y)$ satisfies the following equation:

$$C(x, y) = (1 + U(x, y)) (x^2 + x^3 y(1 + U(x, 1)) + xy (C(x, 1) - x^2(1 + U(x, 1)))) ,$$

which gives the result. \square

Corollary 15. *The g.f. for the popularity of catastrophes in $\mathcal{A}_n^\#$ equals*

$$\frac{x^3(1 - x)(x^2 - x + 1)}{(2x - 1)(x^2 + x - 1)},$$

and for $n \geq 4$ the coefficient of x^n is $3 \cdot 2^{n-4} + 2F_{n-3}$, where F_n is the n -th Fibonacci number (see the sequence A175657 in [20]).

Remark 3. *As a byproduct of Corollary 14 and Corollary 15, the ratio of the popularity of catastrophes in $\mathcal{A}_n^\#$ to the popularity of returns in $\mathcal{A}_n^\#$ tends to $\frac{3}{4}$ when n tends toward ∞ .*

4.5 The number of pyramids

Theorem 14. *For $k \geq 1$, let $P_k(x, y) = \sum_{n, m \geq 0} p_{n, m}^k x^n y^m$ be the g.f. where $p_{n, m}^k$ is the number of n -length non-decreasing Dyck paths with air pockets with m occurrences of the pattern $\Delta_k = U^k D_k$. Then the following holds:*

$$P_k(x, y) = \frac{x^2 \left(1 - \frac{x^2}{1-x} + x^{k-1} \left(1 - x - \frac{x^2(2-x)}{1-x} \right) (y-1) - x^{2k}(y-1)^2 \right)}{\left(1 - x - \frac{x^2}{1-x} - x^{k+1}(y-1) \right) \left(1 - \frac{x^2}{1-x} - x^{k+1}(y-1) \right)}.$$

Proof. First, let us determine the expression of the bivariate g.f. $Z_k(x, y)$ with respect to the length and the number of patterns Δ_k for non-decreasing Dyck paths with air pockets having all their valleys at height 0. The second-to-last return decomposition of \mathcal{A}^\nearrow yields:

$$Z_k(x, y) = (1 + Z_k(x, y)) \left(x^{k+1}y + \left(\frac{x^2}{1-x} - x^{k+1} \right) \right).$$

Hence, we get $Z_k(x, y) = \frac{1}{1 - \frac{x^2}{1-x} - x^{k+1}(y-1)} - 1$.

Now, assuming $k \neq 1$, any Dyck path has one of the following forms: (i) βUD , (ii) $\beta \Delta_{k-1}^\sharp$, (iii) $\beta \Delta_k^\sharp$, (iv) $\beta(\alpha \Delta_k)^\sharp$ with α having all of its valleys at height 0, (v) $\beta \alpha^\sharp$ with $\alpha \in \mathcal{A}^\nearrow$, $\alpha \neq \Delta_{k-1}, \Delta_k, \gamma \Delta_k$ (γ having all of its valleys at height 0), and where β is either empty or has all its valleys at height 0. This yields (for short we use P_k and Z_k instead of $P_k(x, y)$ and $Z_k(x, y)$):

$$P_k = (1 + Z_k) \left(x^2 + x^{k+1}y + x^{k+2}(1 + Z_k) + x \left(P_k - x^k - x^{k+1}y(1 + Z_k) \right) \right).$$

Solving for P_k , we get the result for $k \geq 2$.

If $k = 1$, the expression of $P_1(x, y)$ is the same as that of the bivariate g.f. associated to the pattern D in \mathcal{A}^\nearrow (given in Theorem 12), because D occurs exactly as often as $UD = \Delta_1$. \square

Corollary 16. *For $k \geq 1$, the g.f. for the popularity $\Delta_{\mathbf{k}}(\mathcal{A}_n^\nearrow)$ of k -pyramids in \mathcal{A}_n^\nearrow is*

$$W_k(x) = \frac{x^{k+1}(1-x)(1-4x+5x^2-2x^3+x^5)}{(1-2x)^2(1-x-x^2)}.$$

In particular, we can see that $\Delta_{\mathbf{1}}(\mathcal{A}_n^\nearrow) = \Delta_{\mathbf{k}}(\mathcal{A}_{n+k-1}^\nearrow)$, which means that there are as many 1-pyramids in \mathcal{A}_n^\nearrow as there are k -pyramids in $\mathcal{A}_{n+k-1}^\nearrow$. An asymptotic approximation of the n -th term is $n \cdot 2^{n-5-k}$.

An immediate consequence of the previous corollary is following one, which is the \mathcal{A}_n^\nearrow -counterpart of Corollary 9.

Corollary 17. *For $k \geq 1$, the g.f. for the popularities $\Delta_{\geq \mathbf{k}}(\mathcal{A}_n^\nearrow)$ and $\Delta_{\leq \mathbf{k}}(\mathcal{A}_n^\nearrow)$ are respectively given by*

$$W_{\geq k}(x) = \frac{x^{k-1}}{1-x} W_1(x) \text{ and } W_{\leq k}(x) = \frac{1-x^k}{1-x} W_1(x),$$

which means that $\Delta_{\leq \mathbf{k}}(\mathcal{A}_{n-k+1}^\nearrow) = \Delta_{\geq \mathbf{k}}(\mathcal{A}_n^\nearrow) - \Delta_{\geq \mathbf{k}}(\mathcal{A}_{n-k}^\nearrow)$.

Pattern	Pattern popularity in \mathcal{A}_n^\uparrow	OEIS
U	1, 2, 5, 13, 32, 76, 176, 400, 896, 1984	A098156
D	1, 0, 2, 3, 7, 15, 33, 72, 157, 341	
Peak	1, 1, 3, 7, 16, 36, 80, 176, 384, 832	A045891
Ret	1, 1, 3, 6, 13, 27, 56, 115, 235, 478	A099036
Cat	0, 1, 1, 4, 8, 18, 38, 80, 166, 342	A175657
Δ_k	$\underbrace{0, \dots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
$\Delta_{\geq k}$	$\underbrace{0, \dots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 28, 61, 133, 290, 631$	New (= v_n)
$\Delta_{\leq k}$	$\Delta_{\leq 1}$ 1, 0, 2, 3, 7, 15, 33, 72, 157, 341	$v_n - v_{n-k}$
	$\Delta_{\leq 2}$ 1, 1, 2, 5, 10, 22, 48, 105, 229, 498	
	$\Delta_{\leq 3}$ 1, 1, 3, 5, 12, 25, 55, 120, 262, 570	
	\vdots	

Table 2: Pattern popularity in \mathcal{A}_n^\uparrow for $2 \leq n \leq 11$.

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