Enumeration of Łukasiewicz paths modulo some patterns

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Abstract

For any pattern α of length at most two, we enumerate equivalence classes of Łukasiewicz paths of length $n \geq 0$ where two paths are equivalent whenever the occurrence positions of α are identical on these paths. As a byproduct, we give a constructive bijection between Motzkin paths and some equivalence classes of Łukasiewicz paths.

Keywords: Łukasiewicz path, Dyck path, Motzkin path, equivalence relation, patterns.

1 Introduction and notations

In the literature, lattice paths are widely studied. Their enumeration is a very active field in combinatorics, and they have many applications in other research domains as computer science, biology and physics [18, 19]. Dyck and Motzkin paths are the most often considered. This is partly due to the fact that they are respectively counted by the famous Catalan and Motzkin numbers (see A000108 and A001006 in the on-line encyclopedia of integer sequences [28]). Almost always, these paths are enumerated according to several parameters and statistics (see for instance [6, 14, 15, 17, 20, 21, 24, 25, 29] for Dyck paths and [4, 5, 7, 8, 16, 22, 26] for Motzkin paths). Also, many one-to-one correspondences have been found between lattice paths and some combinatorial objects such as Young tableaux, pattern avoiding permutations, bargraphs, RNA shapes and so on [30]. Recently a new approach has been introduced for studying statistics on lattice paths. It consists in determining the cardinality of the quotient set generated by an equivalence relation based on the positions of a given pattern: *two paths belong to the same equivalence class whenever the positions of* *occurrences of a given pattern are identical on these paths.* Enumerating results are provided for the quotient sets of Dyck, Motzkin and Ballot paths for patterns of length at most three (see respectively [2], [3] and [13]). The purpose of the present paper is to extend these studies for Łukasiewicz paths that naturally generalizes Dyck and Motzkin paths. As a byproduct, we show how Motzkin paths are in one-to-one correspondence with some equivalence classes of Łukasiewicz paths.

Throughout this paper, a *lattice path* is defined by a starting point $P_0 = (0,0)$, an ending point $P_n = (n,0)$, it consists of steps lying in $S = \{(1,i), i \in \mathbb{Z}\}$, and it never goes below the x-axis. The *length* of a path is the number of its steps. We denote by ϵ the empty path, *i.e.*, the path of length zero. Constraining the steps to lie into $\{(1,1),(1,-1)\}$ (resp. $\{(1,1),(1,0),(1,-1)\}$), we retrieve the well known definition of Dyck paths (resp. Motzkin paths). *Łukasiewicz paths* are obtained when the steps belong to $\{(1,i) \in S, i \geq -1\}$. We refer to [10, 23, 30, 32, 33] for some combinatorial studies on Łukasiewicz paths. Let $\mathcal{L}_n$, $\mathcal{D}_n$, $\mathcal{M}_n$, $n \geq 0$, respectively, be the sets of Łukasiewicz, Dyck and Motzkin paths of length n, and $\mathcal{L} = \cup_{n\geq 0}\mathcal{L}_n$, $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$, $\mathcal{M} = \cup_{n\geq 0}\mathcal{M}_n$. For convenience, we set $D = (1,-1)$, $F = U_0 = (1,0)$, $U = U_1 = (1,1)$ and $U_i = (1,i)$ for $i \geq 2$. See Figure 1 for an illustration of Dyck, Motzkin and Łukasiewicz paths of length 18. Note that Łukasiewicz paths can be interpreted as an algebraic language of words $w \in \{x_0, x_1, x_2, \dots\}^\star$ such that $\delta(w) = -1$ and $\delta(w') \geq 0$ for any proper prefix w' of w where δ is the map from $\{x_0, x_1, x_2, \dots\}^\star$ to $\mathbb{Z}$ defined by $\delta(w_1 w_2 \dots w_n) = \sum_{i=1}^{n} \delta(w_i)$ with $\delta(x_i) = i - 1$ (see [11, 27]).

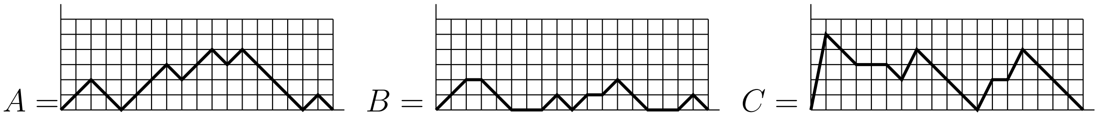

Figure 1: From left to right, we show a Dyck path $A = UUDDUUUDUUDUDDDDUD$, a Motzkin path $B = UUFDDFFUDUFUDDFFUD$ and a Łukasiewicz path $C = U_5DDFFDU_2DDDDU_2FU_2DDDD$.

Any non-empty Łukasiewicz path $L \in \mathcal{L}$ can be decomposed (see [9]) into one of the two following forms: (1) $L = FL'$ with $L' \in \mathcal{L}$, or (2) $L = U_kL_1DL_2D \dots L_kDL'$ with $k \geq 1$ and $L_1, L_2, \dots, L_k, L' \in \mathcal{L}$ (see Figure 2).

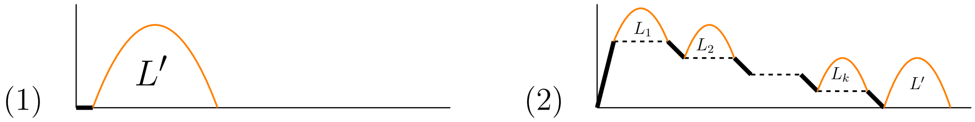

Figure 2: The two forms of the decomposition of a non-empty Łukasiewicz path.

Due to this decomposition, the generating function $L(x)$ for the cardinalities of the sets $\mathcal{L}_n$, $n \geq 0$, satisfies the functional equation $L(x) = 1 + xL(x) + \sum_{k\geq 1} x^{k+1}L(x)^{k+1}$, or equivalently, $L(x) = \frac{1}{1-xL(x)}$. Then, $L(x) = \frac{1-\sqrt{1-4x}}{2x}$ and the coefficient of x^n in the series expansion of $L(x)$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ (see sequence A000108 in [28]).

A *pattern* of length one (resp. two) in a lattice path L consists of one step (resp. two consecutive steps). We will say that an *occurrence* of a pattern is at *position* $i \geq 1$ in L, whenever the first step of this occurrence appears at the i-th step of the path. The *height* of an occurrence is the minimal ordinate reached by its points. For instance, the path $C = U_5DDF\mathbf{FD}U_2DDDDU_2FU_2DDDD$ (see Figure 1) contains one occurrence of the pattern FD at position 5 and of height 2.

Following the recent studies [2, 3, 13], we define an equivalence relation on the set $\mathcal{L}$ for a given pattern α: *two Łukasiewicz paths of the same length are α-equivalent whenever the occurrences of the pattern α appear at the same positions in the two paths.* For instance, $UFFF\mathbf{FD}UDUDFFFFUDFF$ is FD-equivalent to the path C in Figure 1 since the only one occurrence of the pattern FD (in boldface) appear at the same position in the two paths. Note that the height of the occurrences of α is not involved in this definition.

In this paper, for any pattern α of length at most two, we consider the above equivalence relation on the set $\mathcal{L}$, and for each of them we provide the cardinality of the quotient set with respect to the length. Three general methods are used:

- (M_0) we prove that any α-equivalence class contains at least one Motzkin path. Using $\mathcal{M}_n \subset \mathcal{L}_n$ for $n \geq 0$, we deduce that the number of α-equivalence classes in $\mathcal{M}_n$ is equal to that of $\mathcal{L}_n$. Since two of the authors have already determined this number for $\mathcal{M}_n$ (see [3]), we can conclude,
- (M_1) we define a set $\mathcal{A}_n$ of representatives of length $n \geq 0$ and we calculate $a_n = |\mathcal{A}_n|$,
- (M_2) we exhibit a one-to-one correspondence between a subset of Łukasiewicz paths (subset of representatives of the classes) and the set of equivalence classes by using combinatorial reasonings, and then, we evaluate algebraically the generating function for this subset.

The paper is organized as follows. In Section 2, we consider the case of patterns α studied using method (M_0), *i.e.*, $\alpha \in \{U, UU, UD, UF, DU, FU\}$. In Section 3, we focus on these ones that can be dealt using method (M_1), *i.e.*, $\alpha \in \{F, D, FD, DF, DD\}$. In Section 4, we complete our study by the remaining cases which are obtained using method (M_2). We refer to Table 1 for an exhaustive list of our enumerative results.

2 Modulo $\alpha \in \{U, UU, UD, UF, DU, FU\}$

In this section, we focus on the patterns that can be dealt using method (M_0).

Lemma 1 *For $n \geq 0$, let L be a Łukasiewicz path in $\mathcal{L}_n$ and $\alpha \in \{U, UU, UD, UF, DU, FU\}$. Then, there exists a Motzkin path $M \in \mathcal{M}_n$ such that M and L are α-equivalent.*

Proof. Let us assume that $\alpha \in \{U, UU, UD\}$. Any non-empty Łukasiewicz path can be decomposed into one of the two following forms: (i) $L = FL'$ with $L' \in \mathcal{L}$, and (ii) $L = U_kL_1DL_2D\ldots L_kDL'$ with $k \geq 1$ and $L_1, L_2, \ldots, L_k, L' \in \mathcal{L}$. For $n \geq 0$, we recursively define a map ϕ from $\mathcal{L}_n$ to $\mathcal{M}_n$ as follows:

Pattern α	Sequence	Sloane	$a_n, 1 \leq n \leq 10$	Method
U	$\binom{n}{\lfloor \frac{n}{2} \rfloor}$	A001405	$1, 2, 3, 6, 10, 20, 35, 70, 126, 252$	M_0
UU	$\frac{1-2\,x+x^2-\sqrt{(x^2+1)(1-3\,x^2)}}{2x(-1+2x-x^2+x^3)}$	A191385	$1, 1, 1, 2, 3, 5, 7, 12, 18, 31$	
UD	Shift of Fibonacci	A000045	$1, 2, 3, 5, 8, 13, 21, 34, 55, 89$	
UF, FU	$\frac{2}{1-2x+\sqrt{1-4x^3}}$	A165407	$1, 1, 2, 3, 4, 7, 11, 16, 27, 43$	
DU	Shift of Fibonacci	A000045	$1, 1, 1, 2, 3, 5, 8, 13, 21, 34$	
F	$2^n - n$	A000325	$1, 2, 5, 12, 27, 58, 121, 248, 503, 1014$	M_1
D	2^{n-1}	A011782	$1, 1, 2, 4, 8, 16, 32, 64, 128, 256$	
FD, DF	Fibonacci	A000045	$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$	
DD	$\frac{1-x}{1-2x+x^2-x^3}$	A005251	$1, 1, 2, 4, 7, 12, 21, 37, 65, 114$	
U_k	Motzkin	A001006	$1, 2, 4, 9, 21, 51, 127, 323, 835, 2188$	M_2
FF	$\frac{1-3x+4x^2-5x^3+7x^4-7x^5+6x^6-3x^7+x^8}{(1-2x+x^2-x^3)(1-x)^2}$	New	$1, 2, 2, 5, 9, 17, 32, 59, 107, 192$	
FU_k, U_kF	$\frac{1-x-\sqrt{1-2x+x^2-4x^3}}{2x^3}$	A023431	$1, 1, 2, 4, 7, 13, 26, 52, 104, 212$	
U_kD	$\frac{1-x-x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^3}$	A292460	$1, 2, 4, 8, 17, 37, 82, 185, 423, 978$	
DU_k	$\frac{1+x+x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x}$	A004148	$1, 1, 1, 2, 4, 8, 17, 37, 82, 185$	

Table 1: Number of α-equivalence classes for Łukasiewicz paths. The last three sequences are recorded in OEIS [28] as generalized Catalan sequences.

$$\begin{cases} \phi(\epsilon) & = \epsilon, \\ \phi(FL') & = F\phi(L'), \\ \phi(UL_1DL') & = U\phi(L_1)D\phi(L'), \\ \phi(U_kL_1DL_2D\dots L_kDL') & = F\phi(L_1)F\phi(L_2)F\dots\phi(L_k)F\phi(L') \qquad \text{for } k \geq 2. \end{cases}$$

Clearly, $\phi(L)$ is a Motzkin path in $\mathcal{M}_n$, and whenever $\alpha \in \{U, UU, UD\}$ the occurrence positions of α in L and $\phi(L)$ are identical. Then, the equivalence class of L contains a Motzkin path $\phi(L)$.

Let us assume that $\alpha = DU$. Any non-empty Łukasiewicz path L can be written as follows:

$$L = K_0 \prod_{i=1}^{r} (DU)^{a_i} K_i$$

with $r \geq 0$, $a_i \geq 1$ for $1 \leq i \leq r$, and where K_i, $0 \leq i \leq r$, are some parts that do not contain any pattern DU. Note that K_0 and K_r necessarily contain at least one step. From $L \in \mathcal{L}_n$, we define the Motzkin path

$$M = UF^{b_0-1} \left(\prod_{i=1}^{r-1} (DU)^{a_i} F^{b_i} \right) (DU)^{a_r} DF^{b_r-1} \in \mathcal{M}_n$$

where $b_i = |K_i|$ for $0 \leq i \leq r$. Since the occurrence positions of DU in L and M are identical, M is a Motzkin path in the same class as L.

Let us assume that $\alpha \in \{FU, UF\}$. Any non-empty Łukasiewicz path L can be written as follows:

$$L = K_0 \prod_{i=1}^{r} \alpha^{a_i} K_i$$

with $r \geq 0$, $a_i \geq 1$ for $1 \leq i \leq r$, and where K_i, $0 \leq i \leq r$, are some parts that do not contain any pattern α. From $L \in \mathcal{L}_n$, we define the Motzkin path

$$M = F^{b_0} \prod_{i=1}^{r} \alpha^{a_i} D^{c_i} F^{b_i - c_i} \in \mathcal{M}_n$$

where $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$ and $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$. Less formally, for i from 0 to r, K_i is replaced with $D^{c_i} F^{|K_i| - c_i}$ where the value c_i is the maximal number of down steps D that can be placed so that M remains a lattice path. This ensures that M has the same occurrence positions of α as L, which means that M is a Motzkin path in the same class as L. □

Using Lemma 1 and the fact that $\mathcal{M} \subset \mathcal{L}$, we directly deduce the following theorem.

Theorem 1 *For $\alpha \in \{U, UU, UD, UF, DU, FU\}$ and $n \geq 0$, the number of α-equivalence classes in $\mathcal{L}_n$ also is that of $\mathcal{M}_n$.*

Since two of the authors have already determined the number of α-equivalence classes in $\mathcal{M}_n$, we refer to their paper [3] for a detailed description of the different proofs, and we report the results in Table 1.

3 Modulo $\alpha \in \{F, D, FD, DF, DD\}$

In this section, we focus on the patterns that can be dealt with method (M_1) which consists in defining a set $\mathcal{A}_n$ of representatives and counting directly $a_n = |\mathcal{A}_n|$.

Theorem 2 *The number of F-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by $2^n - n$ (see sequence A000325 in [28]).*

Proof. We define the set $\mathcal{A}_n$ as the set of Łukasiewicz paths of length n where the first non-F step (if any) is of the form U_i, $i \geq 1$, and the rest non-F steps are D steps. Since $\mathcal{A}_n$ is in one-to-one correspondence with binary words of length n without a single one, we deduce that $a_n = 2^n - n$. □

Theorem 3 *The number of D-equivalence classes in $\mathcal{L}_n$, $n \geq 1$, is given by 2^{n-1} (see sequence A011782 in [28]).*

Proof. We define the set $\mathcal{A}_n$ of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of non-D steps are F's. Since $\mathcal{A}_n$ is in one-to-one correspondence with binary words of length $n-1$, we deduce that $a_n = 2^{n-1}$. $\square$

Theorem 4 *The number of FD-equivalence (resp. DF-equivalence) classes in $\mathcal{L}_n$, $n \geq 0$, is given by the Fibonacci number f_n defined by $f_0 = 1$, $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (see sequence A000045 in [28]).*

Proof. For the pattern FD (the pattern DF is similar), we define the set $\mathcal{A}_n$, $n \geq 3$, as the set of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of the steps are either F or D steps provided that every D step is immediately preceded by an F step. Clearly, $\mathcal{A}_n$ is decomposed into two sets of paths starting with U_iFF or with U_iFD, which are counted by a_{n-1} and a_{n-2} respectively. $\square$

Theorem 5 *The number of DD-equivalence classes in $\mathcal{L}_n$, $n \geq 0$, is given by the n-th term g_n of the sequence defined by $g_0 = 1$, $g_1 = 1$, $g_2 = 1$, $g_3 = 2$ and $g_n = g_{n-1} + g_{n-2} + g_{n-4}$ for $n \geq 4$ (see sequence A005251 in [28]).*

Proof. We define the set $\mathcal{A}_n$, $n \geq 4$, as the set of Łukasiewicz paths of length n where the first step is of the form U_i, $i \geq 0$, and the rest of the steps are either F or D steps and there are no isolated D steps. Clearly, $\mathcal{A}_n$ is decomposed into sets of paths starting with U_iF, or with U_iDDDF (together with the path U_3DDD for $n = 4$), or with U_iD^j, $j \geq 2, j \neq 3$ which are counted by a_{n-1}, a_{n-4} and a_{n-2} respectively. $\square$

4 Other patterns

In this section, we extend the definition of α-equivalence whenever α is a set S of patterns: two paths L and L' are S-equivalent if for any pattern $\alpha \in S$, the occurrence positions of α are the same in L and L'. We investigate the cases where S is $\{U_k, k \geq 1\}$, $\{DU_k, k \geq 1\}$, $\{U_kD, k \geq 1\}$, $\{FU_k, k \geq 1\}$, and $\{U_kF, k \geq 1\}$. For short, these S-equivalences will be written U_k-equivalence (resp. DU_k-, U_kD-, FU_k-, U_kF-equivalence). Also, we study the FF-equivalence relation in $\mathcal{L}$. For all these cases, we use the method (M_2) that consists in exhibiting subsets of representatives of equivalence classes, and determining algebraically their cardinalities.

4.1 Modulo $S = \{U_k, k \geq 1\}$

Let $\mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height. For instance, we have $U_3DDDFUD \in \mathcal{B}$ and $U_3FDDDUD \notin \mathcal{B}$. Let $\bar{\mathcal{B}} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps.

Lemma 2 *There is a bijection between $\mathcal{B}$ and the set of U_k-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{B}$ (with the same length as L) such that L and L' are equivalent. We write

$$L = K_0 \prod_{i=1}^{r} \alpha_i K_i$$

with $r \geq 0$, where K_i is a part that does not contain any up steps for $0 \leq i \leq r$, and $\alpha_i \in \{U_k, k \geq 1\}$ for $1 \leq i \leq r$. From $L \in \mathcal{L}$, we define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \alpha_i D^{c_i} F^{b_i - c_i}$$

with $b_0 = |K_0|$, and for $1 \leq i \leq r$, $b_i = |K_i|$, $c_i = \min\{b_i, a_i + \sum_{j=1}^{i-1}(a_j - c_j)\}$ where $\alpha_i = U_{a_i}$. Less formally, for i from 0 to r, K_i is replaced with $D^{c_i} F^{|K_i| - c_i}$ where c_i is the maximal number of down steps that can be placed so that L' remains a Łukasiewicz path. Clearly, L' belongs to $\mathcal{B}$ (it does not contain any flat at positive height), and for any $k \geq 1$ the occurrence positions of U_k are the same as for L, *i.e.*, $L' \in \mathcal{B}$ is in the same class as L. For instance, if $L = U_3DUDFFFFUUDDFFDDFF$, then we obtain $L' = U_3DUDDDFFUUDDFFFFFF$ (see Figure 3 for an illustration of this example).

Since the positions of the up steps U_k, $k \geq 1$, remain fixed inside a class, and any flat of $L' \in \mathcal{B}$ lies necessarily on the x-axis, there are no other paths in $\mathcal{B}$ in the same class as L. $\square$

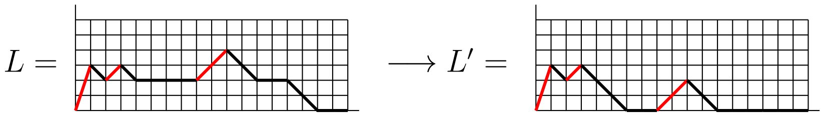

Figure 3: Illustration of the example described in the proof of Lemma 2.

Theorem 6 *The generating function for the set of U_k-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

which generates the Motzkin numbers (A001006 in [28]).

Proof. Using Lemma 2, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Łukasiewicz path $L \in \mathcal{B}$ can be written either $L = FL'$ where $L' \in \mathcal{B}$, or $L = U_k L_1 D L_2 D \ldots L_k D L'$ for $k \geq 1$ where $L_1, L_2, \ldots, L_k \in \bar{\mathcal{B}}$ are some Łukasiewicz paths without flats, and $L' \in \mathcal{B}$. So we obtain the functional equation $B(x) = 1 + xB(x) + xB(x) \sum_{k \geq 1} x^k \bar{B}(x)^k$ where $\bar{B}(x)$ is the generating function for the set $\bar{\mathcal{B}}$ of Łukasiewicz paths without flats. Using the classical decomposition of a Łukasiewicz path, $\bar{B}(x)$ satisfies $\bar{B}(x) = 1 + \sum_{k \geq 2} x^k \bar{B}(x)^k$, or equivalently $\bar{B}(x) = \frac{1}{(1+x)(1-x\bar{B}(x))}$. A simple calculation provides the result. $\square$

Let us define recursively a map ψ from $\mathcal{L}$ to the set of Motzkin paths $\mathcal{M}$ as follows:

$$\begin{cases} \psi(\epsilon) & = \epsilon, \\ \psi(FL) & = F\psi(L), \\ \psi(U_kL_1DL_2D\dots L_kDL) & = U\psi(L_1)F\psi(L_2)F\dots\psi(L_k)D\psi(L), \end{cases}$$

where $L, L_1, L_2, \dots, L_k$ are are some Łukasiewicz paths. See Figure 4 for an illustration of the bijection ψ. For instance, the image by ψ of $U_4FU_2DFDDDU_2U_1DDDFDDFU_2FDU_2DDD$ is $UFUFFDFFUUDFDFFDFUFFUFDD$. Obviously, the map ψ preserves the length of the paths.

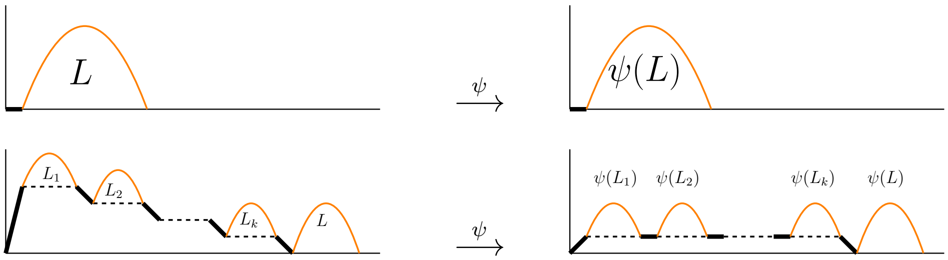

Figure 4: Illustration of the map ψ from $\mathcal{L}$ to $\mathcal{M}$.

We easily deduce the two following facts.

Fact 1 *If $L, L' \in \mathcal{L}$, then $LL' \in \mathcal{L}$ and we have $\psi(LL') = \psi(L)\psi(L')$.*

Fact 2 *There is a one-to-one correspondence between:*

(a) steps $\{U_k, k \geq 1\}$ in L and steps U in $\psi(L)$;

(b) $\{U_kU_\ell, k, \ell \geq 1\}$ in L and UU in $\psi(L)$;

(c) steps F on the x-axis in L and steps F on the x-axis in $\psi(L)$;

(d) $\{U_kD, k \geq 2\} \cup \{U_kF, k \geq 1\}$ in L and UF in $\psi(L)$;

(e) peaks UD in L and peaks UD in $\psi(L)$.

Theorem 7 *For any $n \geq 0$, the map ψ induces a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$.*

Proof. We proceed by induction on n. Obviously, for $n = 0$ we have $\psi(\epsilon) = \epsilon$. We assume that ψ is a bijection from $\mathcal{B}_k$ to $\mathcal{M}_k$, $0 \leq k \leq n$, and we prove the result for $n + 1$. Using the enumerating result of Theorem 6, it suffices to prove that ψ is surjective. So, let M be a Motzkin path in $\mathcal{M}_{n+1}$. We distinguish two cases:

(i) $M = FM'$ with $M' \in \mathcal{M}_n$. Using the recurrence hypothesis, there is $L' \in \mathcal{B}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL'$ lies into $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$ which proves that M belongs to the image by ψ of $\mathcal{B}_{n+1}$.

(ii) $M = UM'DM''$ where M' and M'' are two Motzkin paths in $\mathcal{M}$. We can uniquely write $M' = M_0 \prod_{i=1}^{r} FM_i$ with $r \geq 0$ and where M_i is a (possibly empty) Motzkin path without flat F on the x-axis. Using the recurrence hypothesis, there are $B_0, B_1, \ldots, B_r \in \mathcal{B}$ such that $\psi(B_i) = M_i$, $0 \leq i \leq r$. Also let $B \in \mathcal{B}$ such that $\psi(B) = M''$. Since B_i (resp. B) belongs to $\mathcal{B}$, it does not contain any flat at positive height. Since $M_i = \psi(B_i)$, Fact 2(c) implies that B_i does not contain any flat on the x-axis. So, B_i does not contain any flat steps. So, let us define

$$L = U_{r+1}\left(\prod_{i=0}^{r} B_i D\right) B.$$

Clearly, L lies in $\mathcal{B}_{n+1}$ and satisfies $\psi(L) = M$; then, M belongs to the image by ψ of $\mathcal{B}_{n+1}$, and the map ψ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is a bijection. □

4.2 Modulo $S = \{U_kD, k \geq 1\}$, and $S = \{DU_k, k \geq 1\}$

Let $\mathcal{C} \subset \mathcal{B}$ be the set of Łukasiewicz paths without any flat steps at positive height and such that any up step U_k, $k \geq 1$, is immediately followed by a down step D. For instance, we have $U_3DDDFUD \in \mathcal{C}$ and $U_3FDDDUD \notin \mathcal{C}$. We set $\bar{\mathcal{C}} = \bar{\mathcal{B}} \cap \mathcal{C}$.

Lemma 3 *There is a bijection between $\mathcal{C}$ and the set of U_kD-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{C}$ (with the same length as L) such that L and L' belong to the same class. We write

$$L = K_0 \prod_{i=1}^{r} \left(U_{k_i} D K_i\right),$$

where $k_i \geq 1$ for $1 \leq i \leq r$, and $K_0, K_1, K_2, \ldots, K_r$, $r \geq 0$, are some parts (possibly empty) without pattern U_kD for any $k \geq 1$.

We define the Łukasiewicz path

$$L' = F^{b_0} \prod_{i=1}^{r} \left(U_{k_i} D D^{a_i} F^{b_i - a_i}\right),$$

with $b_i = |K_i|$, $0 \leq i \leq r$, and for $1 \leq i \leq r$, $a_i = \min\{b_i, k_i - 1 + \sum_{j=1}^{i-1}(k_j - 1 - a_j)\}$. Less formally, for i from 0 to r, K_i is replaced with $D^{a_i}F^{b_i-a_i}$ where the value a_i is the maximal number of down steps that can be placed between the two occurrences $U_{k_i}D$ and $U_{k_{i+1}}D$ so that L' remains a lattice path. Clearly, $L' \in \mathcal{C}$ and L' belongs to the same class as L.

For instance, from $L = U_3DUDFFFFUUDDFFDDFF$, we obtain the path $L' = U_3DUDDDFFFUDFFFFFFF$ (see Figure 5 for an illustration of this example).

Now we will prove that any U_kD-equivalence class contains at most one element in $\mathcal{C}$. For a contradiction, let L and L' be two different Łukasiewicz paths in $\mathcal{C}$ belonging to the same class. We write $L = QR$ and $L' = QS$ where R and S start with two different steps. Since L and L' lie in the same class, the two first steps of R and S cannot be U_kD for $k \geq 1$. Moreover, since L (resp. L') lies into $\mathcal{C}$, the two first steps of R (resp. S) cannot constitute a pattern U_kF for $k \geq 1$. Then, R and S cannot start with any up step U_k, $k \geq 1$.

Without loss of generality, let us assume that the first step of R is a down step D and then, the first step of S is a flat step F. This means that the last point of Q has its ordinate equal to zero (otherwise L' could not belong to $\mathcal{C}$). As the first step of R is D, the height of this step is -1 which gives a contradiction and completes the proof. □

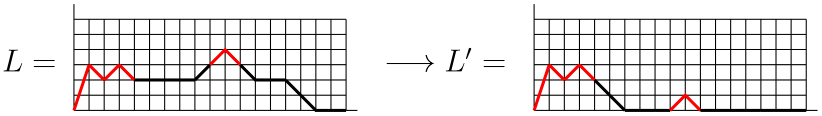

Figure 5: Illustration of the example described in the proof of Lemma 3.

Theorem 8 *The generating function for the set of U_kD-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^3},$$

which generates the sequence A292460 in [28] that is a shift of the generalized Catalan sequence defined by $g_0 = 1$, and $g_{n+1} = g_n + \sum_{k=1}^{n-1} g_k g_{n-1-k}$ for $n \geq 0$ (see A004148 in [28]).

Proof. Using Lemma 3, it suffices to obtain the generating function $C(x)$ for the set $\mathcal{C}$. A non-empty Łukasiewicz path $L \in \mathcal{C}$ can be written either $L = FL'$ where $L' \in \mathcal{C}$, or $L = U_kDL_1DL_2D \ldots L_{k-1}DL'$ for $k \geq 1$ where $L_1, L_2, \ldots, L_{k-1}$ are some Łukasiewicz paths without flats in $\bar{\mathcal{C}}$, and $L' \in \mathcal{C}$. So we obtain the functional equation $C(x) = 1 + xC(x) + x^2C(x)\sum_{k\geq 0} x^k \bar{C}(x)^k$ where $\bar{C}(x)$ is the generating function for the set $\bar{\mathcal{C}}$ of Łukasiewicz paths without flats in $\mathcal{C}$. On the other hand, every $L \in \mathcal{C}$ is decomposed uniquely according to the number $k \geq 0$ of its flat steps as follows: $L = L_0 \prod_{i=1}^{k}(FL_i)$, where $L_i \in \bar{\mathcal{C}}$ for $i \geq 0$. Then by adding a peak $U_{k+1}D$ at the beginning of L and changing each F to D, we obtain a unique element of $\bar{\mathcal{C}}\backslash\{\epsilon\}$, viz. $U_{k+1}DL_0\prod_{i=1}^{k}(DL_i)$. So, we have $\bar{C}(x) - 1 = x^2C(x)$ and a simple calculation provides the result. □

Theorem 9 *For any $n \geq 0$, the map ψ induces a bijection from $\mathcal{C}_n$ to the set of Motzkin paths in $\mathcal{M}_n$ that avoid the pattern UU.*

Proof. Theorem 7 ensures that ψ is a bijection from $\mathcal{B}_n$ to $\mathcal{M}_n$ for $n \geq 0$. We have $\mathcal{C} \subset \mathcal{B}$, and the paths in $\mathcal{C}$ are those in $\mathcal{B}$ that avoid the patterns U_kU_ℓ, $k, \ell \geq 1$ and U_kF, $k \geq 1$.

Using Fact 2(b), the map ψ transforms occurrences of U_kU_ℓ, $k, \ell \geq 1$, into occurrences of UU. Since a path in $\mathcal{C}$ avoids any occurrence of U_kF for $k \geq 1$, the image by ψ of $\mathcal{C}_n$ is the subset of Motzkin paths in $\mathcal{M}_n$ that avoid any occurrence of the pattern UU. □

Let $\mathcal{L}'_n$, $n \geq 2$, be the set of Łukasiewicz paths of length n starting by U and ending by D. For $n \geq 0$, we define the bijection θ from $\mathcal{L}_n$ to $\mathcal{L}'_{n+2}$ as follows: $\theta(L)$ is obtained from L by replacing any occurrence U_kD by an occurrence DU_k for $k \geq 1$, and by adding a step U at the beginning and a step D at the ending. It is straightforward to verify that θ induces a bijection $\bar{\theta}$ between the set of U_kD-equivalence classes of $\mathcal{L}_n$ and the set of DU_k-equivalence classes of $\mathcal{L}'_{n+2}$. Then, Theorem 10 is deduced from Theorem 8.

Theorem 10 *The generating function for the set of DU_k-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 + x + x^2 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x},$$

which generates the generalized Catalan sequence defined by $u_0 = u_1 = 1$, and for $n \geq 2$ $u_n = g_{n-2}$ where g_n is defined in Theorem 8. (see A004148 in [28]).

4.3 Modulo $S = \{U_kF, k \geq 1\}$, and $S = \{FU_k, k \geq 1\}$

Let $\mathcal{E} \subset \mathcal{L}$ be the set of Łukasiewicz paths such that any up step U_k, $k \geq 1$, is immediately followed by a flat step F, and any flat step F of positive height belongs to a pattern U_kF, $k \geq 1$. For instance, we have $U_3FDDFDUD \in \mathcal{E}$ and $U_3DDDFUFD \notin \mathcal{E}$. Let $\bar{\mathcal{E}} \subset \mathcal{E}$ be the set of Łukasiewicz paths in $\mathcal{E}$ without flat step on the x-axis.

Lemma 4 *There is a bijection between $\mathcal{E}$ and the set of U_kF-equivalence classes of $\mathcal{L}$.*

Proof. The proof is obtained *mutatis mutandis* as for Lemma 3 by replacing U_kD with U_kF. □

Theorem 11 *The generating function for the set of U_kF-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - x - \sqrt{1 - 2x + x^2 - 4x^3}}{2x^3},$$

which generates the generalized Catalan sequence defined by $h_0 = 1$, and for $n \geq 0$, $h_{n+1} = h_n + \sum_{k=0}^{n-2} h_k h_{n-2-k}$ (see A023431 in [28]).

Proof. Using Lemma 4, it suffices to obtain the generating function $E(x)$ for the set $\mathcal{E}$. A non-empty Łukasiewicz path $L \in \mathcal{E}$ can be written either $L = FL'$ where $L' \in \mathcal{E}$, or $L = U_kFL_1DL_2D \dots L_kDL'$ for $k \geq 1$ where $L_1, L_2, \dots, L_k$ are some Łukasiewicz paths in $\bar{\mathcal{E}}$, and $L' \in \mathcal{E}$. So we obtain the functional equation $E(x) = 1 + xE(x) + x^3E(x)\sum_{k\geq 0} x^k \bar{E}(x)^{k+1}$ where $\bar{E}(x)$ is the generating function for the set $\bar{\mathcal{E}}$. Using the classical decomposition of a

Łukasiewicz path, we have $\bar{E}(x) = 1 + \sum_{k\geq 3} x^k \bar{E}(x)^{k-1}$. A simple calculation provides the result. $\square$

Let ξ be the map from $\mathcal{L}$ to himself where $\xi(L)$ is obtained from L by replacing any occurrence U_kF by an occurrence FU_k for $k \geq 1$. It is straightforward to verify that ξ induces a bijection $\bar{\xi}$ between the set of U_kF-equivalence classes and the set of FU_k-equivalence classes. Then, we have Theorem 12.

Theorem 12 *The generating function for the set of FU_k-equivalence classes of $\mathcal{L}$ with respect to the length also is the generating function given in Theorem 11.*

Theorem 13 *For $n \geq 0$, the map ψ is a bijection from $\mathcal{E}_n$ to the subset $\mathcal{M}'_n$ of Motzkin paths in $\mathcal{M}_n$ that avoid UU and UD.*

Proof. We proceed by induction on n. Obviously, for $n = 0$, we have $\psi(\epsilon) = \epsilon$. For $0 \leq k \leq n$, we assume that ψ is a bijection from $\mathcal{E}_k$ to the subset $\mathcal{M}'_k$ and we prove the result for $n+1$. Since the set of length n Motzkin paths avoiding UU and UD is enumerated by the value h_n defined in Theorem 11 (see A023431 in [28]), it suffices to prove that ψ is surjective. So, let M be a Motzkin path in $\mathcal{M}'_{n+1}$. We distinguish two cases:

(i) $M = FM'$ with $M' \in \mathcal{M}'_n$. Using the recurrence hypothesis, there is $L' \in \mathcal{E}_n$ such that $M' = \psi(L')$. So, the Łukasiewicz path $L = FL' \in \mathcal{E}_{n+1}$ satisfies $\psi(L) = M$ which proves that M belongs to the image by ψ of $\mathcal{E}_{n+1}$.

(ii) $M = UM'DM''$ where M' and M'' are two Motzkin paths in $\mathcal{M}'$. Since $M \in \mathcal{M}'_{n+1}$, we have $M' \neq \epsilon$ and M' does not start with U, which implies that M' starts with F. Using the recurrence hypothesis, there are $L' \in \mathcal{E}$ and $L'' \in \mathcal{E}$ such that $\psi(L') = M'$ and $\psi(L'') = M''$. Since M' starts with a flat step, L' also starts with a flat step. So, $L = UL'DL''$ belongs to $\mathcal{E}_{n+1}$ and satisfies $\psi(L) = M$ which proves that ψ from $\mathcal{B}_n$ to $\mathcal{M}_n$ is bijective. $\square$

4.4 Modulo FF

Let $\mathcal{F}$ be the set consisting of the union of $\{\epsilon, F\}$ with the set of Łukasiewicz paths containing at most one up step U_k, $k \geq 1$ and such that any flat step F is contained into a pattern FF. For instance, $FFU_3DFFDDFFDFF \in \mathcal{F}$ and $FFU_3FFDDFDFFF \notin \mathcal{F}$.

Lemma 5 *There is a bijection between $\mathcal{F}$ and the set of FF-equivalence classes of $\mathcal{L}$.*

Proof. Let L be a non-empty Łukasiewicz path in $\mathcal{L}$. Let us prove that there exists a Łukasiewicz path $L' \in \mathcal{F}$ (with the same length as L) such that L and L' belong to the same class. We write

$$L = K_1 F^{a_1} K_2 F^{a_2} K_3 \ldots K_r F^{a_r} K_{r+1},$$

with $r \geq 0$ and $a_i \geq 2$ for $1 \leq i \leq r$, such that $K_1, K_2, \ldots, K_r, K_{r+1}$ are some parts without pattern FF, $k \geq 1$, and $K_2, \ldots, K_r$ are not empty and do not have any F in first and last position, and K_1 has no flat in last position, and K_{r+1} has no flat in first position.

If $L = F^n$ with $n \geq 0$, then its equivalence class is reduced to a singleton. Now let us assume that $L \neq F^n$. We distinguish two cases:

(1) K_1 is not empty. We define the Łukasiewicz path

$$L' = U_b D^{b_1-1} F^{a_1} D^{b_2} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}$$

with $b_i = |K_i|$, $1 \leq i \leq r+1$, and $b = \sum_{i=1}^{r+1} b_i - 1$.

(2) K_1 is empty which means that $L = F^{a_1} K_2 F^{a_2} K_3 \ldots K_r F^{a_r} K_{r+1}$. We define the Łukasiewicz path

$$L' = F^{a_1} U_b D^{b_2-1} F^{a_2} D^{b_3} \ldots D^{b_r} F^{a_r} D^{b_{r+1}}$$

with $b_i = |K_i|$, $2 \leq i \leq r+1$, and $b = \sum_{i=2}^{r+1} b_i - 1$.

Less formally, we obtain L' from L by replacing any K_i (excepted the first) with a run of down steps $D^{|K_i|}$, and by replacing the first K_i (K_1 or K_2 according to the case (1) or (2)) with $U_b D^{|K_1|-1}$ (or $U_b D^{|K_2|-1}$) where the up step U_b balances all down steps in L', *i.e.*, b is the number of down steps in L'. Clearly, $L' \in \mathcal{F}$ and L' belongs to the same class as L. For instance, if $L = U_2DFFU_2DDFFU_3DDFFDDFF$, then $L' = U_9DFFDDDFFDDDFFDDFF$ (see Figure 6).

The definition of $\mathcal{L}$ implies that there is only one path of $\mathcal{F}$ in the same class as L, which completes the proof. □

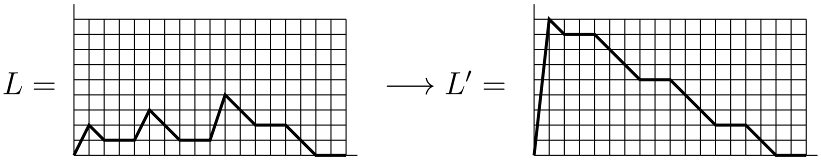

Figure 6: Illustration of the example described in the proof of Lemma 5.

Theorem 14 *The generating function for the set of FF-equivalence classes of $\mathcal{L}$ with respect to the length is given by*

$$\frac{1 - 3x + 4x^2 - 5x^3 + 7x^4 - 7x^5 + 6x^6 - 3x^7 + x^8}{(1 - 2x + x^2 - x^3)(1 - x)^2}.$$

(Note that the associated sequence does not yet appear in [28]).

Proof. Using Lemma 5, it suffices to obtain the generating function $F(x)$ for the set $\mathcal{F}$. A non-empty Łukasiewicz path $L \in \mathcal{F}$ can be written either (i) $L = F^k$ for $k \geq 0$, or (ii) $L = F^{i_0} U_k F^{i_1} D^{j_1} F^{i_2} D^{j_2} \ldots F^{i_\ell} D^{j_\ell} F^{i_{\ell+1}}$ with $\ell \geq 1$, $i_0 = 0$ or $i_0 \geq 2$, $i_1 = 0$ or $i_1 \geq 2$, $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, $i_m \geq 2$ for $2 \leq m \leq \ell$, and $j_m \geq 1$ for $1 \leq m \leq \ell$.

The generating function for the Łukasiewicz paths satisfying (i) is given by $\frac{1}{1-x}$.

For Łukasiewicz paths satisfying (ii), we give the generating function for each part of L, and we multiply them:

- For F^{i_0}, with $i_0 = 0$ or $i_0 \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $F^{i_{\ell+1}}$, with $i_{\ell+1} = 0$ or $i_{\ell+1} \geq 2$, the generating function is $1 + \frac{x^2}{1-x}$;
- For $U_k F^{i_1} D^{j_1}$, with $i_1 = 0$ or $i_1 \geq 2$ and $j_1 \geq 1$, the generating function is $x(1+\frac{x^2}{1-x})\frac{x}{1-x}$;
- For $F^{i_2} D^{j_2} \ldots F^{i_\ell} D^{j_\ell}$, with $i_m \geq 2$, and $j_m \geq 1$, the generating function is $\frac{1}{1-\frac{x^3}{(1-x)^2}}$.

Considering all these cases, we deduce:

$$F(x) = \left(1 + \frac{x^2}{1-x}\right)^3 x^2 \left(1-x\right)^{-1} \left(1 - \frac{x^3}{\left(1-x\right)^2}\right)^{-1} + \left(1-x\right)^{-1}$$

which completes the proof. □

5 Concluding remarks

Extending recent works on Dyck and Motzkin paths [2, 3], the goal of this paper is to calculate the number of Łukasiewicz paths modulo the positions of a given pattern, *i.e.* the number of possible sets $I = \{i_1, i_2, \ldots, i_k\}$ where $i_1, i_2, \ldots, i_k$ are the occurrence positions of the pattern in Łukasiewicz paths. Can one do the same study for other lattice paths such as meanders, bridges and excursions, or Schroeder and Riordan paths?

From our study, we can deduce a lower bound for the maximal cardinality of a class by calculating the average of cardinalities of the classes, *i.e.*, the total number of Łukasiewicz paths divided by the number of classes. Is it possible to calculate the exact value of the maximal cardinality for a class, and for which set I it is reached? Also, it would be interesting to study some properties of the number of Łukasiewicz paths (of a given length) having I as set of positions of the pattern. One can think this number is a polynomial with respect to the length n. If this is true, then we could give properties of these coefficients and roots, which would be a counterpart for lattice paths of the study of descent polynomial on the symmetric group S_n (see MacMahon [12]).

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