EMERGING CONSECUTIVE PATTERN AVOIDANCE

NATHANAËL HASSLER AND SERGEY KIRGIZOV

Abstract. In this note we study the *asymptotic popularity*, that is, the limit probability to find a given consecutive pattern at a random position in a random permutation in the eighteen classes of permutations avoiding at least two length 3 consecutive patterns. We show that for ten classes, this popularity can be readily deduced from the structure of permutations. By combining analytical and bijective approaches, we study in details two more involved cases. The problem remains open for five classes.

1. Introduction and notation

We write a permutation $\pi \in \mathcal{S}_n$ as a word $\pi = a_1 \dots a_n$ whose letters are $\{a_1, \dots, a_n\} = \{1, \dots, n\}$. A *pattern* p of length r is an element of $\mathcal{S}_r$. We usually say that π contains an occurrence of the pattern p if there exists a subsequence $1 \leq i_1 < \dots < i_r \leq n$ such that $a_{i_1} \dots a_{i_r}$ is order-isomorphic to p. In this note we focus on *consecutive* patterns. We say that π contains a consecutive occurrence of the pattern p if there exists a subsequence of consecutive letters $a_i a_{i+1} \dots a_{i+r-1}$ of π that is order-isomorphic to p. We say that π *avoids* the consecutive pattern p if it does not contain any consecutive occurrence of p.

Kitaev [17], along with Mansour [18, 19], presented the enumeration of classes of permutations avoiding at least two length 3 consecutive patterns. In this work, we study the *asymptotic popularity*, that is, the limit probability to find a given pattern of size 3 at random position in a random permutations in the eighteen avoidance classes from Kitaev-Mansour works. We show that, in certain cases, some of the remaining patterns disappear asymptotically. It is a quite enchanting fact.

In the wide realm of interesting papers on permutation patterns, we would like to highlight those that we believe are most relevant to our work. Bóna and Homberger [1, 2, 3] studied the same problem for classical patterns (i.e. non consecutive) in classes of permutations avoiding one length 3 pattern. In particular they proved that among the permutations avoiding the classical pattern 123, and the ones avoiding 132, the decreasing pattern 321 appears asymptotically with probability 1, while the four other length 3 patterns disappear.

Janson [15, 16] considered limit laws for the distributions of classical (non-necessary consecutive) patterns of length 3 in permutations avoiding classical patterns 132 and 321. Borga [6] introduced a method based on generating trees to study asymptotic normality of consecutive pattern occurrences in permutations avoiding certain non-necessary consecutive patterns.

Elizalde and Noy [9] presented a method based on increasing trees and box product [11] to study distributions and avoidance of certain consecutive patterns in permutations. The same authors, in another paper [10], showed how the Goulden-Jackson cluster method [12, 13] can be adapted to enumerate permutations that avoid consecutive patterns.

Barnabei, Bonetti, Silimbani [5] studied joint distributions of consecutive patterns of size 3 in the set of permutations avoiding a non-necessary consecutive pattern 312. They did this by observing how the patterns are transformed by Krattenthaler's bijection [14] between such permutations and

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Université Bourgogne Europe, LIB UR 7534, F-21000 Dijon, France
E-mail addresses: `nathanael.hassler@ens-rennes.fr`, `sergey.kirgizov@u-bourgogne.fr`.
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Class \ Pattern	123	132	213	231	312	321
1 (simple, Sec. 2)			1/2	1/2		
2 (simple, Sec. 2)				0		1
3 (simple, Sec. 2)	1/2					1/2
4 (simple, Sec. 2)			N/A		N/A	
5 (simple, Sec. 2)	1			0		
6 (simple, Sec. 2)				1/2	1/2	
7 (done in [4])				1/2	1/2	0
8 (simple, Sec. 2)			0		0	1
9 (simple, Sec. 2)	1/2				0	1/2
10 (open)			?	?		?
11 (Section 3)			1/4	1/2	1/4	
12 (open)		?	?			?
13 (open)		?	?		?	?
14 (open)	?	?			?	?
15 (open)	?			?	?	?
16 (simple, Sec. 2)		1/4	1/4	1/4	1/4	
17 (Section 4)			1/4	1/2	1/4	0
18 (simple, Sec. 2)	1/2		0		0	1/2

TABLE 1. The asymptotic popularity patterns among eighteen avoidance classes.

Dyck paths, and how Deutsch's involution [8] on Dyck paths helps this process. Their pattern transfer method is similar to what Baril, Burstein and Kirgizov did in their work about faro words and permutations [4]. Their paper is a precursor to the article you are holding in your hands.

The *reverse* $R(\pi)$ of a permutation $\pi = a_1 \dots a_n$ is the permutation $a_n \dots a_1$. The *complement* $C(\pi)$ is the permutation $(n+1-a_1)\dots(n+1-a_n)$. Also $R \circ C$ is the composition of R and C. For example, $R(35214) = 41253$, $C(35214) = 31452$ and $R \circ C(35214) = 25413$. Those bijections preserve the occurrences of consecutive patterns, indeed for $T \in \{R, C, R \circ C\}$, $\pi \in \mathcal{S}_n$ and a consecutive pattern p, π has an occurrence of p if and only if $T(\pi)$ has an occurrence of $T(p)$. As explained in [17], this enables us to focus only on equivalence classes of the avoidance classes under the action of those 3 bijections R, C and $R \circ C$.

Table 1 summarizes our results, presenting eighteen classes, as they appear in [17]. In this table, an empty cell means that corresponding patterns are avoided *by design*, while 0 says that the respective pattern disappears asymptotically (the probability to find this pattern at a random position in a random permutation tends to 0, as the permutation size grows). The values 1/2 and 1/4 present in this table should also be understood in the asymptotic sense. There are two "N/A" for Class 4, because this class is empty for $n > 3$. Question marks indicate open problems.

For consecutive patterns $p_1, \dots, p_k$ and $n \in \mathbb{N}$, we denote by $\mathrm{Av}_n(p_1, \dots, p_k)$ the set of permutations of size n that avoid each of the consecutive patterns $p_1, \dots, p_k$, and by $\mathrm{Av}(p_1, \dots, p_k)$ the set of all permutations that avoid $p_1, \dots, p_k$.

Let $\mathcal{A}_n := \mathrm{Av}_n(p_1, \dots, p_m)$. For a pattern $p \notin \{p_1, \dots, p_m\}$, we denote by $\mathbf{p}_n^{\mathcal{A}}$ the *popularity* of the pattern p in the class $\mathcal{A}_n$, that is, the total number of occurrences of p in $\mathcal{A}_n$. Now we define the *asymptotic popularity* of p in the class $\mathcal{A}$ by

$$\textsc{pop}_{\mathcal{A}}(p) := \lim_{n\to\infty} \frac{\mathbf{p}_n^{\mathcal{A}}}{n|\mathcal{A}_n|}, \tag{1.1}$$

when the limit exists. We will use $\textsc{pop}_k$, where k refers to Class k from Table 1. When it is clear from the context, we simply write $\textsc{pop}$ instead of $\textsc{pop}_{\mathcal{A}}$,

The note is organized as follows. In Section 2 we expose some classes for which asymptotic popularities are easily obtained. In Sections 3 and 4 we compute these popularities for two more complex classes. The methods combine analytic arguments regarding the generating functions and a bijective argument connecting the permutations from the class to involutions. Finally, in Section 5 we offer a conjecture and formulate several questions for future research.

2. Simple classes

In this section, we sum up the classes for which the asymptotic popularities are easily computable. This is the case for classes 1,2,3,5,6,8,9,16 and 18 (see Table 1). Note that the problem is not defined for Class 4, $\mathrm{Av}_n(123, 132, 231, 321)$, since the class is empty as soon as $n > 3$. Class 7, $\mathrm{Av}_n(123, 132, 213)$, has been done by Baril, Burstein and Kirgizov [4, Remark 3.8], using a bijection between such permutations and dispersed Dyck paths that transfers patterns between these two sets of combinatorial objects in a nice and handy way. Their result was the original motivating point for the present work.

Consider Class 1, $\mathrm{Av}_n(123, 132, 312, 321)$ contains only 2 permutations when $n > 1$. These permutations have a simple alternating structure shown at Figure 1. It it clear that asymptotic popularities of the two remaining patterns are equal, $\textsc{pop}(213) = \textsc{pop}(231) = 1/2$.

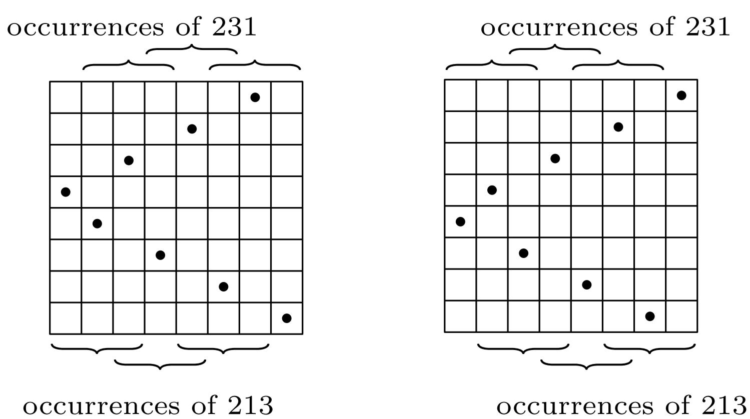

Figure 1. The only 2 permutations of Class 1, $\mathrm{Av}_n(123, 132, 312, 321)$ for $n = 8$.

Class 2, $\mathrm{Av}_n(123, 132, 213, 312)$, also contains just 2 permutations when $n > 1$. The pattern 231 may appear only once, at the very beginning of a permutation. After that, we observe exclusively the occurrences of 321 (for $n > 3$), so $\textsc{pop}(231) = 0$ and $\textsc{pop}(321) = 1$.

Class 3, $\mathrm{Av}_n(132, 213, 231, 312)$, for $n > 1$ consists of two permutations: $123\ldots n$ and $n(n-1)\ldots 321$, thus $\textsc{pop}(123) = \textsc{pop}(321) = 1/2$.

Class 4, $\mathrm{Av}_n(123, 132, 231, 321)$, is empty for $n > 3$.

A permutation from Class 5, $\mathrm{Av}_n(132, 213, 312, 321)$, may have at most one occurrence of pattern 231, so $\textsc{pop}(231) = 0$ and $\textsc{pop}(123) = 1$. There are $n - 1$ permutations in Class 5, for $n > 2$.

For $n > 3$, any permutation from Class 6, $\mathrm{Av}_n(123, 132, 213, 321)$, is a sequence of overlaps of two alternating patterns 231 and 312. We have $\textsc{pop}(231) = \textsc{pop}(312) = 1/2$.

For $n > 3$, Class 8, $\mathrm{Av}_n(123, 132, 231)$, consists of n permutations. Any permutation from this class starts with a sequence of descents. At the end it may have one occurrence of pattern 213 or 312. It follows that $\textsc{pop}(213) = \textsc{pop}(312) = 0$ and $\textsc{pop}(321) = 1$.

A typical permutation from Class 9, $\mathrm{Av}_n(132, 213, 231)$, starts with a sequence of descents, have one occurrence of pattern 312, ends with a sequence of ascents. Permutations $123 \ldots n$ and $n(n-1) \ldots 321$ are also authorized. So, $\textsc{pop}(123) = \textsc{pop}(321) = 1/2$ and $\textsc{pop}(312) = 0$.

Class 16, $\mathrm{Av}_n(123, 321)$, can be solved directly with a symmetry argument. Indeed, this class is stable under the action of the 3 bijections R, C and $R \circ C$. So, any occurrence of the pattern 132 in this class is uniquely mapped to an occurrence of 231 in the same class through R, an occurrence of 312 through C, and an occurrence of 213 through $R \circ C$. Hence $\textsc{pop}_{16}(132) = \textsc{pop}_{16}(231) = \textsc{pop}_{16}(312) = \textsc{pop}_{16}(213) = 1/4$.

Let us describe in detail the case of Class 18, $\mathrm{Av}_n(132, 231)$.

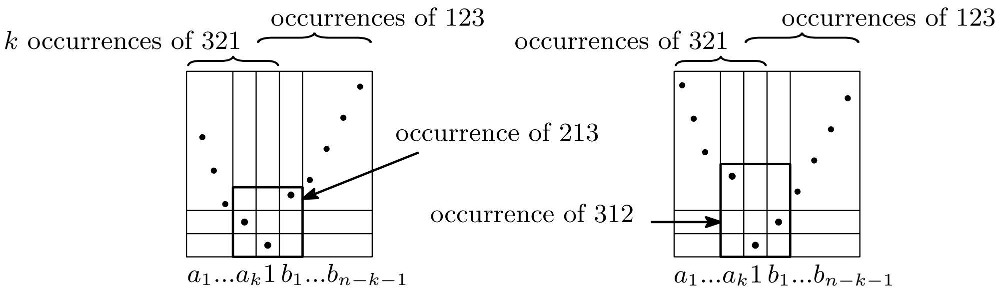

Figure 2. General structure of a permutation from $\mathrm{Av}_n(132, 231)$.

As shown by Kitaev [17], in this case we have the permutations of the form $a_1 \ldots a_k 1 b_1 \ldots b_{n-k-1}$, where $a_1 \ldots a_k$ is a decreasing sequence, and $b_1 \ldots b_{n-k-1}$ an increasing sequence (see Figure 2). In such a permutation, there are exactly $k-1$ occurrences of the pattern 321 (except for $k = 0$, where there is no occurrence), so

$$\mathbf{321}_n = \sum_{k=1}^{n-1} \binom{n-1}{k} (k-1) = (n-1) \cdot 2^{n-2} - 2^{n-1} + 1.$$

Since length n permutations in Class 18 are enumerated by 2^{n-1}, we have

$$\textsc{pop}_{18}(321) = \lim_{n \to \infty} \frac{\mathbf{321}_n}{n \cdot 2^{n-1}} = \frac{1}{2}.$$

Similarly we conclude that $\textsc{pop}_{18}(123) = 1/2$, and $\textsc{pop}_{18}(213) = \textsc{pop}_{18}(312) = 0$.

3. Avoiding 123, 132 and 321

We consider the asymptotic relative popularity of the patterns 213, 231 and 312 in $A_n := \mathrm{Av}_n(123, 132, 321)$ (Class 11 in [17]). From [17, Theorem 3], we know that

$$|A_n| = (n-1)!! + (n-2)!!,$$

where $n!!$ is defined by $0!! = 1$, and for $n \geq 1$

$$n!! = \begin{cases} n \cdot (n-2) \ldots 3 \cdot 1 & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \ldots 4 \cdot 2 & \text{if } n \text{ is even.} \end{cases}$$

For $p \in \{213, 231, 312\}$ and $n \in \mathbb{N}$, let $\mathbf{p}_n$ denote the total number of occurrences of $\mathbf{p}$ in the permutations of $\mathrm{Av}_n(123, 132, 321)$. For $n \geq 2$, there are $n-2$ occurrences of a length 3 pattern in one permutation, so we have $\mathbf{213}_n + \mathbf{231}_n + \mathbf{312}_n = (n-2)|A_n| = (n-2)((n-1)!! + (n-2)!!)$.

Theorem 3.1.

$$\mathbf{231}_n = (n-1)!! \left\lceil \frac{n-3}{2} \right\rceil + (n-2)!! \left\lceil \frac{n-2}{2} \right\rceil,$$

$$\mathbf{312}_n = (n-1)!! \left(\frac{(-1)^{n-1}+n-3}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq n \mod 2}}^{n-1} \frac{1}{k} \right) + (n-2)!! \left(\frac{(-1)^{n}+n-4}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k = n \mod 2}}^{n-2} \frac{1}{k} \right),$$

$$\mathbf{213}_n = (n-2)((n-1)!! + (n-2)!!) - \mathbf{231}_n - \mathbf{312}_n.$$

Proof. From [17, Theorem 3], we know that a permutation $\pi \in A_n$ is *alternating* or *reverse alternating*, namely $\pi = a_1 \ldots a_n$ with $a_1 > a_2 < a_3 \ldots$ or $a_1 < a_2 > a_3 \ldots$. Moreover, for such a permutation we have either $a_n = 1$, or $a_{n-1} = 1$, and if we go from the right to the left starting from 1 and jumping over one element, then we get an increasing sequence. Let A_n^r (resp. A_n^l) be the subset of A_n consisting of permutations π such that $a_n = 1$ (resp. $a_{n-1} = 1$). Again from the proof of [17, Theorem 3], we have $|A_n^r| = (n-2)!!$ and $|A_n^l| = (n-1)!!$. From the structure of those permutations, it is easy to see that the positions of the occurrences of 231 are exactly the indexes $n-2, n-4, \ldots, 2$ or 1 for the permutations of A_n^r, and the indexes $n-3, n-5, \ldots, 2$ or 1 for the permutations of A_n^l. The result on $\mathbf{231}_n$ follows.

Now let us consider $\mathbf{312}_n$. Let $\mathbf{312}_n^r$ (resp. $\mathbf{312}_n^l$) denote the total number of occurrences of the consecutive pattern 312 in A_n^r (resp. A_n^l).

Lemma 3.2. *For any* $n \geq 5$,

(1) $\mathbf{312}_n^r = \mathbf{312}_{n-1}^l$,
(2) $\mathbf{312}_n^l = (n-1)(\mathbf{312}_{n-2}^l + (n-3)!!) - (n-5)!!\frac{(n-3)(n-2)}{2}$.

Proof. (1) follows from the fact that if $\pi \in A_n^r$, then the permutation induced by the $n-1$ first letters of π is in A_{n-1}^l (otherwise we would have an occurrence of 312 on the last 3 letters of π). For (2) consider for $2 \leq k \leq n$ the subset $B_n^k \subset A_n^l$ consisting of the permutations of A_n^l with k in the last position. First assume that $k > 2$, and let $\pi \in B_n^k$. Then $\pi = a_1 \ldots a_{n-2} 2 a_{n-3} 1 k$ with $a_1 \ldots a_{n-3} \in \{3, \ldots, n\} \backslash \{k\}$. The permutation $\pi' := a_1 \ldots a_{n-2} 2 a_{n-3}$ belongs to A_{n-2}^l. We have even that $\pi \longrightarrow \pi'$ is a bijection from B_n^k to A_{n-2}^l, that preserves the number of occurrences of 312 on the first $n-2$ letters. So for each occurrence of 312 in a permutation of A_{n-2}^l, we have an occurrence of 312 in the first $n-2$ letters of a permutation of B_n^k. It remains to count the number of occurrences of 312 in the last 4 letters of each permutation of B_n^k, that is in $2a_{n-3}1k$ with our notation. Note that $2a_{n-3}1$ can never be an occurrence of 312, and $a_{n-3}1k$ is one only if $a_{n-3} > k$. Consequently, for each permutation of A_{n-2}^l we have an extra occurrence of 312 in a permutation of B_n^k that does not end by $2b1k$ with $3 \leq b < k$. Since there are $(k-3)|A_{n-4}^l|$ such permutations, the total number of occurrences of 312 in B_n^k is $\mathbf{312}_{n-2}^l + |A_{n-2}^l| - (k-3)|A_{n-4}^l|$. Similarly, when $k = 2$, the total number of occurrences of 312 in B_n^2 is $\mathbf{312}_{n-2}^l + |A_{n-2}^l|$. We finally deduce that

$$\begin{aligned} \mathbf{312}_n^l &= \mathbf{312}_{n-2}^l + |A_{n-2}^l| + \sum_{k=3}^{n} (\mathbf{312}_{n-2}^l + |A_{n-2}^l| - (k-3)|A_{n-4}^l|) \\ &= (n-1)(\mathbf{312}_{n-2}^l + (n-3)!!) - (n-5)!!\frac{(n-3)(n-2)}{2}. \end{aligned}$$

□

Let $u_n := \frac{\mathbf{312}^l_n}{(n-1)!!}$ and $f(z) := \sum_{n=3}^{\infty} u_n z^n$. From Lemma 3.2 (2) we deduce that $u_n = u_{n-2} + 1 - \frac{n-2}{2(n-1)}$ for any $n \geq 5$. We easily derive that

$$f(z) = \frac{z(2(z-1)\ln(1-z) + z^3 + 3z^2 - 2z)}{4(1-z)^2(1+z)},$$

and finally

$$\mathbf{312}^l_n = (n-1)!! \left(\frac{(-1)^{n-1} + n - 3}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq n \mod 2}}^{n-1} \frac{1}{k} \right),$$

and with Lemma 3.2 (1),

$$\mathbf{312}^r_n = (n-2)!! \left(\frac{(-1)^{n} + n - 4}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k = n \mod 2}}^{n-2} \frac{1}{k} \right).$$

□

Corollary 3.3. $\textsc{pop}_{11}(231) = 1/2$, *and* $\textsc{pop}_{11}(213) = \textsc{pop}_{11}(312) = 1/4$.

4. Avoiding 123, 132

In [7], Claesson proved that the Foata transform induces a bijection between $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$, the set of involutions of size n, showing in particular that $|\mathrm{Av}_n(123, 132)| = |\mathcal{I}_n|$. Let us recall briefly this process. An involution $\pi \in \mathcal{I}_n$ has cycles of length 1 or 2. We introduce a standard form for writing π:

(1) Each cycle is written with its least element first.
(2) The cycles are written in decreasing order of their least element.

Denote by $\hat{\pi}$ the permutation obtained from π by writing it in standard form and by erasing the parentheses separating the cycles. Then $\pi \mapsto \hat{\pi}$ is a bijection between $\mathcal{I}_n$ and $\mathrm{Av}_n(123, 132)$. We will use this bijection to study the frequencies of each pattern of length 3 in $\mathrm{Av}_n(123, 132)$, taking advantage of some knowledge we have on the involutions. Table 2 shows the correspondence between patterns in $\mathrm{Av}_n(123, 132)$ and in $\mathcal{I}_n$.

Pattern in $\mathrm{Av}_n(123, 132)$	Pattern in $\mathcal{I}_n$, with $a < b < c$	Fixed point-free pattern in $\mathcal{I}_n$
321	$(c)(b)(a)$ or $(c)(b)(a\ \star)$ or $(\star\ c)(b)(a)$	$\emptyset$
231	$(b\ c)(a)$ or $(b\ c)(a\ \star)$	$(b\ c)(a\ \star)$
213	$(b)(a\ c)$ or $(\star\ b)(a\ c)$	$(\star\ b)(a\ c)$
312	$(c)(a\ b)$ or $(\star\ c)(a\ b)$	$(\star\ c)(a\ b)$

Table 2. The correspondence between patterns in $\mathrm{Av}_n(123, 132)$ and $\mathcal{I}_n$.

Example 4.1. *The involution* $\pi = 732458169 \in \mathcal{I}_9$ *has standard form* $\pi = (9)(6\ 8)(5)(4)(2\ 3)(1\ 7)$, *so* $\hat{\pi} = 968542317 \in \mathrm{Av}_9(123, 132)$. *The occurrence 854 of the pattern* 321 *in* $\hat{\pi}$ *corresponds to the occurrence* $(6\ 8)(5)(4)$ *of the pattern* $(\star\ c)(b)(a)$ *in* π.

Lemma 4.2. *Let* $\textsc{fp}_n$ *be the total number of fixed points in* $\mathcal{I}_n$. *Then*

$$\frac{\mathrm{FP}_n}{|\mathcal{I}_n|} \underset{n\to\infty}{\sim} \sqrt{n}.$$

Proof. The generating function $I(z,u)$ such that the coefficient of $z^n u^k$ is the number of involutions in $\mathcal{I}_n$ having k fixed points, divided by $n!$, is given by $I(z,u) = e^{zu+z^2/2}$. The average number of fixed points in $\mathcal{I}_n$ is then

$$\frac{\mathrm{FP}_n}{|\mathcal{I}_n|} = \frac{[z^n]\partial_u I(z,u)|_{u=1}}{[z^n]I(z,1)} = \frac{[z^{n-1}]I(z,1)}{[z^n]I(z,1)}.$$

It is known, see for instance [11, 21], that

$$[z^n]I(z,1) \underset{n\to\infty}{\sim} \frac{n^{-n/2}}{2\sqrt{\pi n}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right),$$

which allows us to conclude after simplification[1]. □

Lemma 4.2 indicates that fixed points are quite rare within the involutions. Consequently, in order to compute the number of occurrences of a pattern in $\mathrm{Av}_n(123, 132)$, it suffices to compute the number of occurrences of the corresponding fixed point-free pattern in $\mathcal{I}_n$.

Lemma 4.3. *Let* $\boldsymbol{\alpha} := (b\ c)(a\ \star)$, $\boldsymbol{\beta} := (\star\ b)(a\ c)$ *and* $\boldsymbol{\gamma} := (\star\ c)(a\ b)$ *be the three consecutive fixed point-free patterns of size 3 in* $\mathcal{I}_n$, *and let respectively* $\boldsymbol{\alpha}_n$, $\boldsymbol{\beta}_n$ *and* $\boldsymbol{\gamma}_n$ *denote their number of occurrences in* $\mathcal{I}_n$. *Then* $\mathrm{POP}_{17}(321) = 0$,

$$\mathrm{POP}_{17}(231) = \lim_{n\to\infty} \frac{\boldsymbol{\alpha}_n}{n|\mathcal{I}_n|}, \quad \mathrm{POP}_{17}(213) = \lim_{n\to\infty} \frac{\boldsymbol{\beta}_n}{n|\mathcal{I}_n|}, \quad \textit{and} \quad \mathrm{POP}_{17}(312) = \lim_{n\to\infty} \frac{\boldsymbol{\gamma}_n}{n|\mathcal{I}_n|}.$$

Proof. Let $\boldsymbol{\delta}_n$ be the total number of occurrences of a pattern admitting a fixed point in $\mathcal{I}_n$. In order to prove the lemma, it suffices to show that $\lim_{n\to\infty} \boldsymbol{\delta}_n/(n|\mathcal{I}_n|) = 0$. Since each fixed point appears in 3 different occurrences of a pattern, we have $\boldsymbol{\delta}_n \leq 3 \cdot \mathrm{FP}_n$. The result follows from Lemma 4.2. □

Proposition 4.4. $\mathrm{POP}_{17}(321) = 0$ *and* $\mathrm{POP}_{17}(231) = 1/2$.

Proof. The first statement has been proved in Lemma 4.3. For the second one, Lemma 4.3 indicates that it suffices to compute the popularity of the pattern $\boldsymbol{\alpha}$ among the fixed point-free patterns in $\mathcal{I}_n$. Such a pattern consists of 2 consecutive transpositions. But for each pair of consecutive transpositions in a permutation of $\mathcal{I}_n$, we have exactly one occurrence of the pattern $\boldsymbol{\alpha}$, and one occurrence of $\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$. Thus, the popularity of $\boldsymbol{\alpha}$ among fixed point-free patterns in $\mathcal{I}_n$ is $\frac{1}{2}$, which finishes the proof. □

Now let us focus on the pattern 213. By Lemma 4.3, it suffices to estimate the popularity of the pattern $(\star\ b)(a\ c)$ in the involutions. By definition of the Foata transform, such an occurrence is necessarily of the form $(b\ c)(a\ d)$, with $a < b < c < d$. In $\mathrm{Av}(123, 132)$, it corresponds to an occurrence of the pattern 2314. Therefore we compute the exponential generating function of $\mathbf{2314}_n$, the number of occurrences of 2314 in $\mathrm{Av}_n(123, 132)$.

Lemma 4.5. $\mathbf{2314}_4 = 1$, $\mathbf{2314}_5 = 4$, *and for all* $n \geq 6$,

$$\mathbf{2314}_n = \mathbf{2314}_{n-1} + (n-1)\mathbf{2314}_{n-2} + \binom{n-2}{2}|\mathcal{I}_{n-4}|.$$

Proof. It is easy to see that each permutation in Class 17 has 1 in either the first (type 1) or the second (type 2) position from the right. The number of occurrences of 2314 in each type 1 permutation of size n is exactly $\mathbf{2314}_{n-1}$. Let $\pi \in \mathrm{Av}_n(123, 132)$ of type 2. Then $\pi = a_1 \ldots a_{n-2}1k$ for some $k \in \{2, \ldots, n\}$, and $a_1 \ldots a_{n-2} \in \mathrm{Av}_{n-2}(123, 132)$. This is in fact a bijection between the type 2 permutations of $\mathrm{Av}_n(123, 132)$ ending with k and $\mathrm{Av}_{n-2}(123, 132)$. So for each occurrence of

[1]See also Michael Lugo's blog post
`http://godplaysdice.blogspot.com/2008/02/how-many-fixed-points-do-involutions.html`

2314 in a permutation of $\mathrm{Av}_{n-2}(123,132)$, we have $n-1$ occurrences of 2314 in a type 2 permutation of $\mathrm{Av}_n(123,132)$. However, occurrences can also appear on the last positions of type 2 permutations. Indeed, there is one more occurrence for each permutation ending with $2i1k$, for $3 \leq i < k \leq n$, and there are $\binom{n-2}{2}|\mathcal{I}_{n-4}|$ such permutations. In the end, we obtain the desired formula. $\square$

Proposition 4.6. *Let $G(z) = \sum_{n=4}^{+\infty} \frac{\mathbf{2314}_n}{n!} z^n$ be the EGF of $(\mathbf{2314}_n)_{n\geq 4}$. Then*

$$G(z) = \frac{e^{\frac{(1+z)^2}{2}}}{2} \int_0^z e^{-\frac{(1+t)^2}{2}} dt + \frac{z(z-2)e^{z+\frac{z^2}{2}}}{4}.$$

Proof. From Lemma 4.5 we can verify that G satisfies the following Cauchy problem:

$$\begin{cases} G''(z) - (1+z)G'(z) - G(z) = \frac{z^2}{2} e^{z+\frac{z^2}{2}}, \\ G(0) = G'(0) = 0. \end{cases}$$

Note that $e^{z+\frac{z^2}{2}}$ is the EGF of $(|\mathcal{I}_n|)_{n\geq 0}$. Solving this differential equation, we get that its only solution is the one stated in the lemma. $\square$

Corollary 4.7. $\mathrm{POP}_{17}(312) = \mathrm{POP}_{17}(213) = 1/4$.

Proof. We proceed in two steps. First we prove that $[z^n]z(z-2)e^{z+\frac{z^2}{2}} \sim \frac{n|\mathcal{I}_n|}{n!}$, and secondly we show that $[z^n]\left(e^{\frac{(1+z)^2}{2}} \int_0^z e^{-\frac{(1+t)^2}{2}} dt\right) = o\left(\frac{n|\mathcal{I}_n|}{n!}\right)$. Those two facts prove that $[z^n]G(z) \sim \frac{n|\mathcal{I}_n|}{4 \cdot n!}$, and so $\mathrm{POP}_{17}(2314) = \mathrm{POP}_{17}(213) = 1/4$. By Proposition 4.4 we then have by deduction $\mathrm{POP}_{17}(312) = 1/4$. For the first point, we already know (see for instance [11, 21]) that

$$[z^n]e^{z+\frac{z^2}{2}} = \frac{|\mathcal{I}_n|}{n!} \underset{n\to\infty}{\sim} \frac{n^{-n/2}}{2\sqrt{\pi n}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right).$$

We deduce after simplification that

$$\frac{|\mathcal{I}_{n-1}|}{(n-1)!} \sim \frac{n^{-n/2}}{2\sqrt{\pi}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right), \text{ and } \frac{|\mathcal{I}_{n-2}|}{(n-2)!} \sim \frac{\sqrt{n} \cdot n^{-n/2}}{2\sqrt{\pi}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right).$$

Consequently,

$$[z^n]z(z-2)e^{z+\frac{z^2}{2}} = \frac{|\mathcal{I}_{n-2}|}{(n-2)!} - 2\frac{|\mathcal{I}_{n-1}|}{(n-1)!} \sim \frac{\sqrt{n} \cdot n^{-n/2}}{2\sqrt{\pi}} \exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right) \sim \frac{n|\mathcal{I}_n|}{n!},$$

which proves the first point. For the second one, let us start by studying the coefficients of the other term of $G(z)$.

Lemma 4.8. *Let $F(z) = e^{\frac{(1+z)^2}{2}} \int_0^z e^{-\frac{(1+t)^2}{2}} dt$. Then for all $n \geq 0$, $[z^n]F(z) \in \frac{1}{n!} \cdot \mathbb{N}$.*

Proof. F satisfies the following equality: $F'(z) = (1+z)F(z) + 1$. Then, with a direct induction, there exist two polynomials $p_n, q_n \in \mathbb{N}[z]$, with $\deg(p_n) = n$ and $\deg(q_n) = n-1$ such that $F^{(n)}(z) = p_n(z)F(z) + q_n(z)$. Since $F(0) = 0$, we have $F^{(n)}(0) \in \mathbb{N}$, and as $[z^n]F(z) = \frac{F^{(n)}(0)}{n!}$, the result follows. $\square$

Remark 4.9. *$F'(z)$ is the EGF of the sequence A000932 in OEIS* [20].

Lemma 4.8 ensures the positivity of the coefficients of $F(z)$. We can then apply a saddle point bound to F (see [11, Corollary VIII.1]). Indeed, by the positivity, $F(z)z^{-n-1}$ has a unique saddle point ζ defined by

$$\zeta \frac{F'(\zeta)}{F(\zeta)} = n+1, \text{ or equivalently } \frac{\zeta}{F(\zeta)} = n+1-\zeta-\zeta^2, \tag{4.1}$$

and then

$$[z^n]F(z) \leq \frac{F(\zeta)}{\zeta^n}. \tag{4.2}$$

Given the saddle point equation (4.1), it seems out of reach to obtain an exact expression of ζ. However, the saddle point is the value minimizing the right-hand term in (4.2), and since we just look for a good enough upper bound on $[z^n]F(z)$, a nice approximation of the saddle point may yield a sufficient bound. It turns out that choosing $\zeta = \sqrt{n}$ is sufficient. The upper bound (4.2) then becomes

$$\begin{aligned}
[z^n]F(z) &\leq \frac{e^{\frac{(1+\sqrt{n})^2}{2}} \int_0^{\sqrt{n}} e^{-\frac{(1+t)^2}{2}} dt}{(\sqrt{n})^n} \\
&\leq \left(\int_0^{+\infty} e^{-t-\frac{t^2}{2}} dt \right) \cdot n^{-n/2} \cdot \exp\left(\frac{n}{2} + \sqrt{n}\right).
\end{aligned}$$

This is indeed enough to prove that $[z^n]F(z) = o\left(\sqrt{n} \cdot n^{-n/2} \cdot \exp\left(\frac{n}{2} + \sqrt{n}\right)\right) = o\left(\frac{n|\mathcal{I}_n|}{n!}\right)$. $\square$

5. Open questions

The conjecture, presented below, is true for cases solved in our work. Is it true in general case?

Conjecture 5.1. *For $m > 0$, denote by $p_1, \ldots, p_k, p$ certain consecutive patterns of length m. Let $\mathcal{A}_n = \mathrm{Av}_n(p_1, \ldots, p_k)$ and $\mathcal{B}_n = \mathrm{Av}_n(p_1, \ldots, p_k, p)$. Suppose that $\mathrm{POP}_{\mathcal{A}}(p) = 0$, then for every consecutive pattern q of length m we have $\mathrm{POP}_{\mathcal{B}}(q) = \mathrm{POP}_{\mathcal{A}}(q)$.*

It may also be interesting to answer the following questions:

(1) How to solve Classes 10, 12, 13, 14 and 15 using an enumerative or probabilistic approach? Our numerical experiments suggest that the convergence rate appears to be quite low to formulate plausible conjectures about these cases.
(2) What happens when we avoid only one consecutive pattern of size 3?
(3) Can we find a set of patterns, avoiding which we will obtain irrational asymptotic popularity for some remaining pattern?
(4) Does the limit (1.1) always exist? If not, can we characterize patterns for which this limit exists?

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