Grand Dyck paths with air pockets

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Abstract

Grand Dyck paths with air pockets (GDAP) are a generalization of Dyck paths with air pockets by allowing them to go below the x-axis. We present enumerative results on GDAP (or their prefixes) subject to various restrictions such as maximal/minimal height, ordinate of the last point and particular first return decomposition. In some special cases we give bijections with other known combinatorial classes.

1 Introduction

In a recent paper [2], the authors introduce and study a new class of lattice paths, called *Dyck paths with air pockets* (*DAP* for short). Such a path is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geqslant 1$, where two down steps cannot be consecutive. See Figure 1 for an example. These paths can be viewed as ordinary Dyck paths where each maximal run of down-steps is condensed into one large down step. As mentioned in [2], they also correspond to a stack evolution with (partial) reset operations that cannot be consecutive (see for instance [5]). In this paper, we generalize these paths to *grand Dyck path with air pockets* (*GDAP* for short), which have the same definition as *DAP*, except that they can go below the x-axis, and the empty path ε is considered as a GDAP.

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The main goal is to make enumerative studies on GDAP (or prefix of these paths) with various restrictions on the maximal height reached, minimal height reached, height of the last point, ...

The remaining of this paper is structured as follows. The next section recalls some useful results from [2] and introduces several notations for particular subsets of GDAP. The main result of Section 3 is Theorem 1 which gives the generating function counting the GDAP with respect to the length, which is the 'grand' counterpart of the generating function in (1) for DAP. In Section 4 we give similar results for prefixes of GDAP ending at a given ordinate, the corresponding problem for DAP being already solved in [10]. In Section 5, we provide generating functions for the number of partial GDAP that never go below the line $y = m$, with respect to the ordinate of the last point. In Section 6, we count partial GDAP lying between the lines $y = 0$ and $y = t$, which correspond to partial DAP bounded by a given height $t > 0$. We present a constructive bijection between these paths of length n for $t = 2$ and the set of compositions of $n - 2$ such that no two consecutive parts have the same parity. In Section 7, we count partial GDAP lying between the lines $y = -t$ and $y = t$, and we present a constructive bijection between these paths of length n for $t = 1$ and the set of compositions of $n + 3$ such that the first part is odd, the last part is even, and no two consecutive parts have the same parity. Finally, in Section 8 we provide enumerative results for DAP with a special first return decompostion, which proves that there are in one-to-one correspondence with Motzkin paths avoiding the patterns UH, HU and HH. We leave as an open question the problem of finding a constructive bijection between these two sets.

2 Definitions and notations

2.1 DAP

The *length* of a path is the number of its steps, and for $n \geqslant 0$, let $\mathcal{A}_n$ be the set of n-length DAP. By definition $\mathcal{A}_0 = \mathcal{A}_1 = \varnothing$ and we set $\mathcal{A} = \bigcup_{n\geqslant 2} \mathcal{A}_n$, see [2]. A DAP is called *prime* whenever it ends with D_k, $k \geqslant 2$, and returns to the x-axis only once. The set of all prime DAP of length n is denoted $\mathcal{P}_n$. Notice that UD is not prime, where for short we denote D_1 by D, so we set $\mathcal{P} = \bigcup_{n\geqslant 3} \mathcal{P}_n$. If $\alpha = U\beta UD_k \in \mathcal{P}_n$, then $2 \leqslant k < n$ and β is a (possibly empty) prefix of a path in $\mathcal{A}$, and we define the DAP $\alpha^\flat = \beta UD_{k-1}$, called the 'lowering' of α. For example, the path $\alpha = UUDUUD_3$ is prime, and $\alpha^\flat = UDUUD_2$. The map $\alpha \mapsto \alpha^\flat$ is clearly a bijection from $\mathcal{P}_n$ to $\mathcal{A}_{n-1}$ for all $n \geqslant 3$, and we denote by $\gamma^\sharp$ the inverse image of $\gamma \in \mathcal{A}_{n-1}$ ($\alpha^\sharp$ is a kind of ‘elevation’ of α).

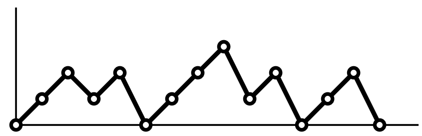

Figure 1: The Dyck path with air pockets $UUDUD_2UUUD_2UD_2UUD_2$.

Any DAP $\alpha \in \mathcal{A}$ can be decomposed depending on its *second-to-last return to the x-axis*: either (i) $\alpha = UD$, or (ii) $\alpha = \beta UD$ with $\beta \in \mathcal{A}$, or (iii) $\alpha \in \mathcal{P}$, or (iv) $\alpha = \beta\gamma$ where $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{P}$. So, if $A(x) = \sum_{n\geqslant 2} a_n x^n$ where a_n is the cardinality of $\mathcal{A}_n$, and $P(x) = \sum_{n\geqslant 3} p_n x^n$ where p_n is the cardinality of $\mathcal{P}_n$, then we have $P(x) = xA(x)$ and the previous decompositions imply the functional equation $A(x) = x^2 + x^2A(x) + xA(x) + xA(x)^2$, and

$$A(x) = \frac{1 - x - x^2 - \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}{2x}, \tag{1}$$

which generates the generalized Catalan numbers (see A004148 in [8]). The first values of a_n for $2 \leqslant n \leqslant 10$ are $1, 1, 2, 4, 8, 17, 37, 82, 185$. In [2], the authors study the enumeration of these paths according to many parameters, and they give a constructive bijection between these paths and peakless Motzkin paths (i.e. lattice paths in the first quadrant, starting at $(0, 0)$, ending on the x-axis, made of steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, and avoiding peaks of the form UD).

2.2 GDAP

The main object of study in this paper are grand Dyck paths with air pockets (GDAP for short) which generalize DAP by allowing such paths to go below the x-axis; and for convenience the empty path ε is a GDAP. See the first path in Figure 2 for an example. Let $\mathcal{G}_n$ be the set of GDAP of length $n \geqslant 0$, and we set $\mathcal{G} = \bigcup_{n\geqslant 0} \mathcal{G}_n$.

We introduce notations for several subsets of $\mathcal{G}$ used in this study:

- $\mathcal{G}^+$ is the set of GDAP starting with U, and the empty path;
  - $\mathcal{G}^+_1$ is the set of elements of $\mathcal{G}^+$ ending with $D_k, k \geqslant 1$;
  - $\mathcal{G}^+_2$ is the set of elements of $\mathcal{G}^+$ ending with U.
- $\mathcal{G}^-$ is the set of GDAP starting with $D_k, k \geqslant 1$;

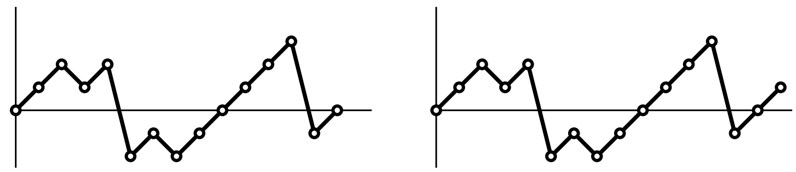

Figure 2: A grand Dyck path with air pockets $UUDUD_4UDUUUUUD_4U$, and a partial GDAP ending at ordinate 1 with an up-step.

- $\mathcal{G}_1^-$ is the set of elements of $\mathcal{G}^-$ ending with $D_k, k \geqslant 1$;
- $\mathcal{G}_2^-$ is the set of elements of $\mathcal{G}^-$ ending with U.

Obviously, we have $\mathcal{G}^+ = \{\varepsilon\} \cup \mathcal{G}_1^+ \cup \mathcal{G}_2^+$, $\mathcal{G}^- = \mathcal{G}_1^- \cup \mathcal{G}_2^-$, and $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$. Also, we denote by $\overleftarrow{\mathcal{P}}$ the set of GDAP obtained from a prime DAP in $\mathcal{P}$ by mirroring it. For instance, the mirror of $U^3D_2UD_2 \in \mathcal{P}$ is $D_2UD_2U^3 \in \overleftarrow{\mathcal{P}}$.

3 Enumeration of GDAP

In this section, we present a generating function that counts GDAP with respect to the length.

Any element of $\mathcal{G}_1^+$ is of the form $\alpha\beta$, where $\alpha \in \mathcal{G}^+$ and $\beta \in \mathcal{P} \cup \{UD\}$. Any element of $\mathcal{G}_2^+$ is either of the form (i) $\alpha\beta$ where $\alpha \in \mathcal{G}_2^+$ and $\beta \in \overleftarrow{\mathcal{P}} \cup \{DU\}$, or of the form (ii) $\overline{\alpha\beta}$ where $\alpha \in \mathcal{G}_1^+$, $\beta \in \overleftarrow{\mathcal{P}} \cup \{DU\}$, and $\overline{\alpha\beta}$ is the path obtained by merging the last step of α together with the first step of β, i.e. if $\alpha = xD_i$ and $\beta = D_jy$, then $\overline{\alpha\beta} = xD_{i+j}y$. Finally, any element of $\mathcal{G}^+$ is either the empty path ε, an element of $\mathcal{G}_1^+$, or an element of $\mathcal{G}_2^+$.

Thus, we deduce the following system of equations:

$$\begin{cases} G_1^+(x) = & G^+(x)(x^2 + P(x)) \\ G_2^+(x) = & G_2^+(x)(x^2 + P(x)) + \dfrac{1}{x} G_1^+(x)(x^2 + P(x)) \\ G^+(x) = & 1 + G_1^+(x) + G_2^+(x), \end{cases}$$

where P, G_1^+, G_2^+, and G^+ are the generating functions with respect to the length for the cardinalities of $\mathcal{P}$, $\mathcal{G}_1^+$, $\mathcal{G}_2^+$, and $\mathcal{G}^+$, respectively. We have $P(x) = xA(x)$, where A is the generating function for the set $\mathcal{A}$ of Dyck paths with air pockets, see (1). Solving the system, we get:

$$G_1^+(x) = \frac{x^2}{\sqrt{x^4-2x^3-x^2-2x+1}},$$
$$G_2^+(x) = \frac{(1-x-x^2)\sqrt{x^4-2x^3-x^2-2x+1}-x^4+2x^3+x^2+2x-1}{2x^4-4x^3-2x^2-4x+2},$$
$$G^+(x) = \frac{(1-x+x^2)\sqrt{x^4-2x^3-x^2-2x+1}+x^4-2x^3-x^2-2x+1}{2x^4-4x^3-2x^2-4x+2},$$

and the first terms of the respective series expansions associated with those generating functions are:

- $x^2+x^3+2x^4+5x^5+11x^6+26x^7+63x^8+153x^9+376x^{10}+O(x^{11})$,
- $x^3+2x^4+5x^5+13x^6+32x^7+80x^8+201x^9+505x^{10}+O(x^{11})$,
- $1+x^2+2x^3+4x^4+10x^5+24x^6+58x^7+143x^8+354x^9+881x^{10}+O(x^{11})$.

They correspond to the OEIS sequences A051286, A110320, and A110236.

On the other hand, any element of $\mathcal{G}^-$ is of the form $\alpha\beta$, where $\alpha \in \overleftarrow{\mathcal{P}} \cup \{DU\}$ and $\beta \in \mathcal{G}$. Any element of $\mathcal{G}$ is either an element of $\mathcal{G}^+$ or an element of $\mathcal{G}^-$.

Thus, we deduce the following system of equations:

$$\begin{cases} G^-(x) = & (x^2+P(x))G(x) \\ G(x) = & G^+(x) + (x^2+P(x))G(x), \end{cases}$$

where G^- and G are the generating functions with respect to the length for the cardinalities of $\mathcal{G}^-$ and $\mathcal{G}$, respectively. Solving the system, we get:

$$G^-(x) = \frac{(1-x+x^2-R)(x^4-2x^3-x^2-2x+1+(1-x+x^2)R)}{2(1+x-x^2+R)R}$$

with $R = \sqrt{x^4-2x^3-x^2-2x+1}$, and we have the following result.

Theorem 1. *The o.g.f. that counts the set $\mathcal{G}$ with respect to the length is given by*

$$G(x) = \frac{x^4-2x^3-x^2-2x+1+(1-x+x^2)R}{(1+x-x^2+R)R}$$

with R defined as above.

Notice that there is a bijection χ between the sets $\mathcal{G}_1^+$ and $\mathcal{G}_2^-$ defined as follows: for $\alpha \in \mathcal{G}_1^+$, $\chi(\alpha)$ is simply the mirror of α, for instance $\chi(UUUD_3) = D_3UUU$.

So, we easily have

$$\begin{cases} G_2^-(x) = & G_1^+(x) \\ G_1^-(x) = & G^-(x) - G_2^-(x). \end{cases}$$

The first terms of the series expansions of G^-, G, G_1^- and G_2^- are respectively

- $x^2 + x^3 + 3x^4 + 7x^5 + 16x^6 + 39x^7 + 95x^8 + 233x^9 + 577x^{10} + O(x^{11})$,
- $1+2x^2+3x^3+7x^4+17x^5+40x^6+97x^7+238x^8+587x^9+1458x^{10}+O(x^{11})$,
- $x^4 + 2x^5 + 5x^6 + 13x^7 + 32x^8 + 80x^9 + 201x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 5x^5 + 11x^6 + 26x^7 + 63x^8 + 153x^9 + 376x^{10} + O(x^{11})$.

They correspond to the OEIS sequences A203611, A051291, A110320, and A051286.

4 Partial GDAP ending at a given ordinate

Let pre($\mathcal{G}$) be the set of partial GDAP, i.e. the set of all prefixes of elements of $\mathcal{G}$, see the second path in Figure 2 for an example. In this part, we enumerate partial GDAP ending at a given ordinate with an up-step (resp. with a down-step) with respect to the length. Let f_k (resp. g_k) be the generating function for the number (with respect to the length) of partial GDAP ending at ordinate $k \in \mathbb{Z}$ with an up-step (resp. with a down-step). For short, we will write f_k and g_k instead of $f_k(x)$ and $g_k(x)$.

According to the results in Section 2, for $k = 0$ we obviously have:

$$f_0 = G_2^+(x) + G_2^-(x) = \frac{1 - x + x^2 + \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}{2\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}} - 1,$$

and

$$g_0 = G_1^+(x) + G_1^-(x) = \frac{\left(1 + x - x^2 - \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}\right)x}{2\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}.$$

The first terms of the series expansions of f_0 and g_0 are respectively

- $x^2 + 2x^3 + 4x^4 + 10x^5 + 24x^6 + 58x^7 + 143x^8 + 354x^9 + 881x^{10} + O(x^{11})$,
- $x^2 + x^3 + 3x^4 + 7x^5 + 16x^6 + 39x^7 + 95x^8 + 233x^9 + 577x^{10} + O(x^{11})$.

They correspond to the sequences A110236 and A203611 in OEIS. Clearly, we have $G(x) = 1 + f_0 + g_0$.

For $k > 0$, a partial GDAP ending at ordinate k can be written $\alpha\beta$, where α is either empty or a GDAP ending on the x-axis with an up-step, and β is a partial DAP ending at ordinate k. Then, we obtain

$$f_k + g_k = (1 + f_0) \cdot T_k(x)$$

with

$$T_k(x) = x^k s_2^{k+1}, \text{ and } s_2 = \frac{1 + x - x^2 - \sqrt{-x^2 - 2x^3 - 2x + x^4 + 1}}{2x},$$

where T_k is the o.g.f. that counts DAP ending at ordinate k with respect to length, which is already obtained in [10].

As a consequence, we deduce the following result.

Theorem 2. *The o.g.f. that counts the partial GDAP ending at a positive ordinate (with respect to the length) is given by*

$$\frac{\left(x^2 - x - 1 + \sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}\right)^2}{4x\sqrt{x^4 - 2\,x^3 - x^2 - 2\,x + 1}}.$$

The first terms of the series expansion are: $x + x^2 + 4x^3 + 9x^4 + 22x^5 + 55x^6 + 136x^7 + 339x^8 + 849x^9 + 2132x^{10} + O(x^{11})$.

For $k < 0$, a partial GDAP ending at ordinate k can be written β or $\overline{\alpha\beta}$, where α is a GDAP ending with a down-step, and β is the symmetric about the x-axis of a partial DAP ending at ordinate $-k > 0$ in the right-to-left model studied in [10]. Then, we obtain

$$f_k + g_k = R_k(x)\left(1 + \frac{g_0(x)}{x}\right),$$

with

$$R_k(x) = (s_2 - 1) \cdot \frac{s_2^{-k-1}}{x}.$$

For instance, the first terms of the series expansion for $k = -1, -2$ are

- $x + 2x^2 + 4x^3 + 10x^4 + 24x^5 + 58x^6 + 143x^7 + 354x^8 + 881x^9 + 2204x^{10} + O(x^{11})$,
- $x + 2x^2 + 5x^3 + 13x^4 + 32x^5 + 80x^6 + 201x^7 + 505x^8 + 1273x^9 + 3217x^{10} + O(x^{11})$ which correspond to the sequences A110236 and A110320 in OEIS.

Notice that partial GDAP of length $n - 1$ and ending at ordinate $k = -1$ are in one-to-one correspondence with non-empty GDAP of length n and starting with an up-step (see the set $\mathcal{G}^+$ from Section 3). To perceive it, one can add an up-step at the end of a partial GDAP ending at $k = -1$, apply a symmetry about the x-axis, and consider the mirror of this path.

Moreover, $x^2 \cdot G_2^+(x)$ (see Section 3) is the generating function for partial GDAP ending at $k = -2$. Such partial GDAP of length n are in one-to-one correspondence with paths of length $n+2$ in $\mathcal{G}_2^+$. To see it, from a partial GDAP ending at $k = -2$, one can add an up-step at the beginning of the path and another one at the end.

Since there is an infinite number of partial paths of length n ending at negative height, we cannot provide an ordinary generating function (with respect to the length) for these paths. So, we get around this in the next section by counting partial GDAP lying above the line $y = m$ for a given $m \leqslant 0$.

5 Minorized partial GDAP

Let us denote by $\text{pre}(\mathcal{G})_m$ the set of partial GDAP which never go below the line $y = m$, $m \leqslant 0$, and let us reuse the same notations as in the previous section for the generating functions f_k and g_k in this subset of $\text{pre}(\mathcal{G})$. Obviously, we have $f_k = 0$ for all $k \leqslant m$ and $g_k = 0$ for all $k < m$. By convenience, we count the empty path in f_0. Then, the o.g.f.'s satisfy the following equations:

$$\begin{cases} & f_0 = 1 + xf_{-1} + xg_{-1}, \\ \forall k \geqslant m+1, k \neq 0, & f_k = xf_{k-1} + xg_{k-1}, \\ \forall k \geqslant m, & g_k = \sum_{i=1}^{\infty} xf_{k+i}. \end{cases}$$

As a consequence, we have $f_{m+1} = xg_m$. Now, we introduce the bivariate generating functions

$$f(u,x) = f(u) = \sum_{k=m+1}^{\infty} u^k f_k \quad \text{and} \quad g(u,x) = g(u) = \sum_{k=m}^{\infty} u^k g_k.$$

Making use of the recursions above, we get:

$$\begin{aligned} f(u) &= 1 + xu\left(f(u) + g(u)\right), \\ g(u) &= \frac{x}{1-u}\left(u^m f(1) - f(u)\right). \end{aligned}$$

Plugging the second equation into the first one, we get:

$$f(u) = 1 + xuf(u) + \frac{x^2 u}{1-u}\left(u^m f(1) - f(u)\right).$$

Solving for $f(u)$, we finally get:

$$f(u) = \frac{1 - u + x^2 u^{m+1} f(1)}{1 - u - xu + xu^2 + x^2 u}. \tag{2}$$

In order to compute $f(1)$, we use the kernel method (see [3, 9]) on $f(u)$. We can rewrite the denominator—which is a polynomial in u, of degree 2—as $x(u - r_1)(u - r_2)$, where:

$$r_1 = \frac{1 + x - x^2 + \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x},$$

$$r_2 = \frac{1 + x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x},$$

and then, relation (2) implies

$$f(u) \cdot (x(u - r_1)(u - r_2)) = 1 - u + x^2 u^{m+1} f(1).$$

Plugging $u = r_2$ (which has a Taylor expansion at $x = 0$), we obtain:

$$1 - r_2 + x^2 r_2^{m+1} f(1) = 0,$$

which gives an expression for $f(1)$:

$$f(1) = \frac{r_2 - 1}{x^2 r_2^{m+1}},$$

and then:

$$f(u) = \frac{1 - u + (r_2 - 1)\left(\frac{u}{r_2}\right)^{m+1}}{1 - u - xu + xu^2 + x^2 u},$$

$$g(u) = \frac{x}{1 - u}\left(u^m f(1) - f(u)\right).$$

Finally, we have:

Theorem 3. *The o.g.f. that counts the partial GDAP above the line $y = m$ (with respect to the length) is given by*

$$f(1) + g(1) = \frac{r_2^{-m} - r_2^{-1-m} - x^2}{x^3}.$$

For instance, if $m = -1, -2$ the first terms of the series expansions are

- $1 + 2x + 4x^2 + 8x^3 + 17x^4 + 37x^5 + 82x^6 + 185x^7 + 423x^8 + 978x^9 + 2283x^{10} + O(x^{11})$,
- $1 + 3x + 6x^2 + 13x^3 + 29x^4 + 65x^5 + 148x^6 + 341x^7 + 793x^8 + 1860x^9 + 4395x^{10} + O(x^{11})$.

They correspond to the sequences A004148 and A093128 in OEIS.

6 Partial (G)DAP bounded by $y = 0$ and $y = t$

6.1 Enumerative results

In this section, we count partial GDAP lying between the lines $y = 0$ and $y = t$, which correspond to partial DAP bounded by a given height $t > 0$. We introduce the notation f_k^t, g_k^t for $0 \leqslant k \leqslant t$, $f^t(u)$, and $g^t(u)$, which are the counterparts of f_k, g_k, $f(u)$, and $g(u)$ defined in the previous section. So, we deduce the following system of equations:

$$\left[\begin{array}{cccc|cccc}
-1 & & & & 0 & & & \\
x & -1 & & & x & 0 & & \\
 & \ddots & \ddots & & & \ddots & \ddots & \\
 & & x & -1 & & & x & 0 \\ \hline
0 & x & \dots & x & -1 & & & \\
 & 0 & \ddots & \vdots & & \ddots & & \\
 & & \ddots & x & & & \ddots & \\
 & & & 0 & & & & -1
\end{array}\right] \cdot \left[\begin{array}{c} f_0^t \\ \vdots \\ \vdots \\ f_t^t \\ \hline g_0^t \\ \vdots \\ \vdots \\ g_t^t \end{array}\right] = \left[\begin{array}{c} -1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right].$$

For a given height $t \geqslant 0$, the previous matrix (denoted A_t) is square with $2(t+1)$ rows. Using classical properties of the determinant (in particular $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$ whenever D is invertible, and C and D commute [7]), we can easily prove that $D_t = \det(A_t)$ satisfies

$$D_{t+2} + (x^2 - x - 1)D_{t+1} + xD_t = 0,$$

anchored with $D_0 = 1$, and $D_1 = 1 - x^2$. Then we deduce

$$D_t = \frac{2^t x^{t+1}}{W} \left(\frac{W - x^2 + x - 1}{(W - x^2 + x + 1)^{t+1}} + (-1)^{t+1} \frac{W + x^2 - x + 1}{(W + x^2 - x - 1)^{t+1}} \right),$$

where

$$W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}.$$

For instance, we have $D_2 = x^4 - x^3 - 2x^2 + 1$, and $D_3 = -x^6 + 2x^5 + 2x^4 - 2x^3 - 3x^2 + 1$.

Using Cramer's rule to solve the system, for $0 \leqslant k \leqslant t$, we have

$$f_k^t = \frac{N_k^t}{D_t}, \quad g_k^t = \frac{N_{t+1+k}^t}{D_t}, \tag{3}$$

where N_k^t is the determinant of the matrix $A_t(k)$ obtained from A_t by replacing the $(k+1)$-th column with the vector $(-1, 0, \ldots, 0)^T$.

As we have done for D_t, it is easy to prove that N_k^t satisfies the following recurrence relations, for $0 \leqslant k \leqslant t$:

$$\begin{cases} N_0^t &= D_t \\ N_{2t+1}^t &= 0 \\ N_k^t &= xN_{k-1}^{t-1} & 1 \leqslant k \leqslant t \\ N_{t+k}^t &= xN_{t+k-2}^{t-1} & 2 \leqslant k \leqslant t \\ N_{t+1}^t &= x^2 N_0^{t-1} + xN_t^{t-1}. \end{cases}$$

See Table 1 for exact values of N_k^t when $0 \leqslant t \leqslant 3$ and $0 \leqslant k \leqslant 7$.

$k \backslash t$	0	1	2	3
0	1	$-x^2+1$	$x^4 - x^3 - 2x^2 + 1$	$-x^6 + 2x^5 + 2x^4 - 2x^3 - 3x^2 + 1$
1	0	x	$-x^3 + x$	$x^5 - x^4 - 2x^3 + x$
2		x^2	x^2	$-x^4 + x^2$
3		0	$-x^4 + x^3 + x^2$	x^3
4			x^3	$x^6 - 2x^5 - x^4 + x^3 + x^2$
5			0	$-x^5 + x^4 + x^3$
6				x^4
7				0

Table 1: The first values of N_k^t for $0 \leqslant t \leqslant 3$ and $0 \leqslant k \leqslant 7$.

Using (3) and the above recurrence relations for N_k^t, we can deduce closed forms for f_k^t, g_k^t, $0 \leqslant k \leqslant t$.

So, we can state the following result.

Theorem 4. *The o.g.f. that counts the nonempty GDAP bounded by the lines $y = 0$ and $y = t$ (with respect to the length) is*

$$g_0^t = \frac{N_{t+1}^t}{D_t}$$

with

$$N_{t+1}^t = \frac{2^{t+2}x^{t+3}(-1)^t}{W(x^2-x-1)^2 - W^3}\left(\frac{1}{(x^2-x-1+W)^t} - \frac{1}{(x^2-x-1-W)^t}\right).$$

For instance, if $t = 1, 2, 3, 4$, then we have

$$g_0^1 = \frac{x^2}{1-x^2}, g_0^2 = \frac{x^2\left(1+x-x^2\right)}{x^4-x^3-2x^2+1}, g_0^3 = \frac{x^2\left(x^4-2x^3-x^2+x+1\right)}{(x^3-2x^2-x+1)\left(1+x-x^3\right)}, \text{ and}$$

$$g_0^4 = \frac{-x^8+3x^7-3x^5-2x^4+x^3+x^2}{x^8-3x^7-x^6+5x^5+4x^4-3x^3-4x^2+1},$$

and the first terms of the series expansion of these generating functions are respectively

- $x^2 + x^4 + x^6 + x^8 + x^{10} + O(x^{11})$,
- $x^2 + x^3 + x^4 + 3x^5 + 2x^6 + 6x^7 + 6x^8 + 11x^9 + 16x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 9x^7 + 22x^8 + 32x^9 + 66x^{10} + O(x^{11})$,
- $x^2 + x^3 + 2x^4 + 4x^5 + 7x^6 + 16x^7 + 27x^8 + 63x^9 + 112x^{10} + O(x^{11})$.

The first two correspond to shifts of A000035 and A062200. The last two sequences do not appear in [8].

6.2 Bijection with a set of compositions

As stated above, the enumeration of the set $\mathcal{G}_n^{[0,2]}$ of GDAP bounded by $y = 0$ and $y = 2$ is given by $g_0^2 = \frac{x^2(1+x-x^2)}{x^4-x^3-2x^2+1}$ which have a series expansion where the coefficients coincide (up to a shift) with the sequence A062200 in [8]. In this part, for any $n \geqslant 0$, we exhibit a constructive bijection ψ between $\mathcal{G}_n^{[0,2]}$ and the set $\mathcal{C}(n-2)$ of compositions of $n-2$ such that no two consecutive parts have the same parity (see [6] for the enumeration of these objects, and [4] for more results about the enumeration of compositions with regard to several statistics on parts).

Let us define the map ψ. Assuming $n \geqslant 2$, let $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{G}_n^{[0,2]}$ and $\alpha' = \alpha_2\alpha_3 \dots \alpha_{n-1}$. We write $\alpha' = B_1B_2 \dots B_r$ where each B_i is a subpath of α' satisfying the following rules:

- if α' does not contain U^2 and UD_2, then $r = 1$ and $B_1 = \alpha'$;
- otherwise, we split α' into subpaths B_i, $1 \leqslant i \leqslant r$, by cutting it after all up-steps that are followed by another up-step or a D_2-step.

For instance, if $\alpha = UUD_2UUDUD_2UDUDUUD_2 = U\alpha'D_2$, then $\alpha' = B_1B_2B_3B_4B_5$ where $B_1 = U$, $B_2 = D_2U$, $B_3 = UDU$, $B_4 = D_2UDUDU$, $B_5 = U$. We refer to Figure 3 for an illustration of this decompostion.

Let $b_1, \dots, b_r$ be the lengths of the subpaths $B_1, \dots, B_r$, respectively. It is clear that $b_1 + b_2 + \dots + b_r = n-2$. Moreover, if the subpath B_i starts with U and ends with U, then B_i is of he form $U(DU)^k$ for some $k \geqslant 0$, and

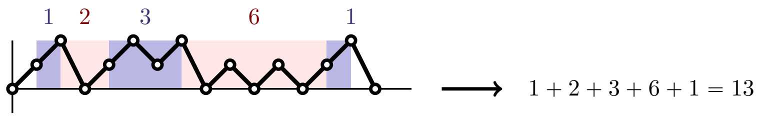

Figure 3: The image by ψ of $\alpha = UUD_2UUDUD_2UDUDUUD_2$ is $\psi(\alpha) = 1, 2, 3, 6, 1$.

B_{i+1} is necessarily of the form $D_2(UD)^\ell U$ for some ℓ, which implies that b_i is odd and b_{i+1} is even; if the subpath B_i starts with D_2 and ends with U, then B_i is of the form $D_2(UD)^k U$ for some $k \geqslant 0$, and B_{i+1} is necessarily of the form $U(DU)^\ell$ for some ℓ, which implies that b_i is even and b_{i+1} is odd. Thus, two consecutive b_i and b_{i+1} always have different parities. Then, the above procedure defines a map ψ from $\mathcal{G}_n^{[0,2]}$ to the set $\mathcal{C}(n-2)$, and for $\alpha \in \mathcal{G}_n^{[0,2]}$, we set $\psi(\alpha) = b_1, b_2, \ldots, b_r$.

Theorem 5. *The map ψ from $\mathcal{G}_n^{[0,2]}$ to $\mathcal{C}(n-2)$ is a bijection.*

Proof. Since $\mathcal{G}_n^{[0,2]}$ and $\mathcal{C}(n-2)$ have the same cardinality (see the o.g.f. g_0^2 and the sequence A062200 at the end of subsection 6.1, it suffices to prove that ψ is surjective.

Let $c = c_1, \ldots, c_r$ be a composition in $\mathcal{C}(n-2)$, with $r \geqslant 2$ (the case $r = 1$ being trivial since if c_1 is even, then we have $c = \psi(U(DU)^{c_1/2}D)$, and if c_1 is odd we have $c = \psi(U(UD)^{(c_1-1)/2}UD_2))$.

For $r \geqslant 2$, we distinguish four cases: (i) c_1 and c_r are even, (ii) c_1 is even and c_r is odd, (iii) c_1 and c_r are odd, (iv) c_1 is odd and c_r is even. According to each case, we define $\alpha \in \mathcal{G}_n^{[0,2]}$ such that $\psi(\alpha) = c$ as follows:
Case (i):

$$\alpha = U\ (DU)^{c_1/2}\ (UD)^{(c_2-1)/2}U\ D_2U(DU)^{(c_3-2)/2} \ldots D_2U(DU)^{(c_r-2)/2}\ D;$$

Case (ii):

$$\alpha = U\ (DU)^{c_1/2}\ (UD)^{(c_2-1)/2}U\ D_2U(DU)^{(c_3-2)/2} \ldots (UD)^{(c_r-1)/2}U\ D_2;$$

Case (iii):

$$\alpha = U\ (UD)^{(c_1-1)/2}U\ D_2U(DU)^{(c_2-2)/2}\ (UD)^{(c_3-1)/2}U \ldots (UD)^{(c_r-1)/2}U\ D_2;$$

Case (iv):

$$\alpha = U\ (UD)^{(c_1-1)/2}U\ D_2U(DU)^{(c_2-2)/2}\ (UD)^{(c_3-1)/2}U \ldots D_2U(DU)^{(c_r-2)/2}\ D.$$

For each case, it is clear that α belongs to $\mathcal{G}_n^{[0,2]}$, which implies that ψ is surjective, and then bijective. $\square$

7 GDAP bounded by $y = -t$ and $y = t$

7.1 Enumerative results

In this section, we count GDAP lying between the lines $y = -t$ and $y = t$. We introduce the notation f_k^t, g_k^t for $-t \leqslant k \leqslant t$, $f^t(u)$, and $g^t(u)$, which are the counterparts of f_k, g_k, $f(u)$, and $g(u)$. So, we deduce the following system of equations:

$$\left[\begin{array}{cccc|cccc} -1 & & & & 0 & & & \\ x & -1 & & & x & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & x & -1 & & & x & 0 \\ \hline 0 & x & \dots & x & -1 & & & \\ & 0 & \ddots & \vdots & & \ddots & & \\ & & \ddots & x & & & \ddots & \\ & & & 0 & & & & -1 \end{array}\right] \cdot \left[\begin{array}{c} f_{-t}^t \\ \vdots \\ f_{-1}^t \\ f_0^t \\ f_1^t \\ \vdots \\ f_t^t \\ \hline g_{-t}^t \\ \vdots \\ \vdots \\ g_t^t \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \end{array}\right].$$

For a given height $t \geqslant 0$, the previous matrix (denoted $A_t^{'}$) is square with $2(2t+1)$ rows. We notice that for all $t \geqslant 0$, the matrix $A_t^{'}$ is identical to the matrix A_{2t} defined in the previous section. Hence, we have $D_t^{'} := \det(A_t^{'}) = \det(A_{2t}) = D_{2t}$, i.e.

$$D_t^{'} = \frac{4^t x^{2t+1}}{W}\left(\frac{W - x^2 + x - 1}{(W - x^2 + x + 1)^{2t+1}} - \frac{W + x^2 - x + 1}{(W + x^2 - x - 1)^{2t+1}}\right),$$

where

$$W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}.$$

Using Cramer's rule to solve the system, for $-t \leqslant k \leqslant t$, we have

$$f_k^t = \frac{\widetilde{N}_k^t}{D_t^{'}}, \quad g_k^t = \frac{\widetilde{N}_{2t+1+k}^t}{D_t^{'}}, \tag{4}$$

where $\widetilde{N}_k^t$ is the determinant of the matrix $A_t'(k)$ obtained from A_t' by replacing the $(k+t+1)$-th column with the vector $(0,\dots,0,-1,0,\dots,0)^T$, where the -1 is in the $(t+1)$-th position.

Now, we focus on the calculation of f_k^t and g_k^t for $k=0$. The other cases can be obtained similarly, but they are much more technical and less interesting to present them here. With the same arguments as in the previous section (in particular, using the mentioned property of the determinant on blocks), it is easy to prove that $\widetilde{N}_0^t$ satisfies:

$$\widetilde{N}_0^t = D_{t-1} \cdot D_t.$$

Moreover, we have

$$\widetilde{N}_{2t+1}^t = D_{t-1} \cdot N_{t+1}^t.$$

Using the results obtained in the previous section, these two relations allow to obtain a close form for $\widetilde{N}_0^t$ and $\widetilde{N}_{2t+1}^t$. Using (4), we deduce close forms for f_0^t and g_0^t and we can state the following result.

Theorem 6. *The o.g.f. that counts the nonempty GDAP bounded by the lines $y=-t$ and $y=t$ (with respect to the length) is*

$$f_0^t + g_0^t = \frac{D_{t-1}}{D_{2t}} \cdot \left(D_t + N_{t+1}^t\right)$$

where D_t and N_{t+1}^t are defined in the previous section.

For instance, if $t=1,2,3$, then we have

$$f_0^1 + g_0^1 = \frac{1}{x^4-x^3-2x^2+1}, \quad f_0^2 + g_0^2 = \frac{(x-1)(x+1)^2}{x^7-2x^6-3x^5+2x^4+6x^3+3x^2-x-1},$$

$$f_0^3 + g_0^3 = \frac{\left(x^4-x^3-2x^2+1\right)^2}{x^{12}-5x^{11}+4x^{10}+10x^9-4x^8-19x^7-4x^6+17x^5+11x^4-5x^3-6x^2+1},$$

and the first terms of the series expansion of these generating functions are respectively

- $1+2x^2+x^3+3x^4+4x^5+5x^6+10x^7+11x^8+21x^9+27x^{10}+O(x^{11})$,
- $1+2x^2+3x^3+5x^4+13x^5+22x^6+48x^7+93x^8+190x^9+375x^{10}+O(x^{11})$,
- $1+2x^2+3x^3+7x^4+15x^5+36x^6+75x^7+176x^8+386x^9+869x^{10}+O(x^{11})$.

The first one corresponds to a shift of A122514. The last two sequences do not appear in [8].

7.2 Bijection with a set of compositions

As stated above, the enumeration of the set $\mathcal{G}_n^{[-1,1]}$ of GDAP bounded by $y = -1$ and $y = 1$ is given by $f_0^1 + g_0^1 = \frac{1}{x^4 - x^3 - 2x^2 + 1}$ which have a series expansion where the coefficients coincide (up to a shift) with the sequence A122514. In this part, for any $n \geqslant 0$, we exhibit a constructive bijection ϕ between $\mathcal{G}_n^{[-1,1]}$ and the set $\mathcal{C}'(n+3)$ of compositions of $n+3$ such that the first part is odd, the last part is even, and no two consecutive parts have the same parity.

Now, let us define the map ϕ. Assuming $n \geqslant 2$, let $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{G}_n^{[-1,1]}$. We write $\alpha = B_1 B_2 \dots B_r$ where each B_i is a subpath of α obtained by applying the same decomposition made on α' in subsection 6.2. Let $b_1, b_2, \dots, b_r$ be the lengths of subpaths $B_1, \dots, B_r$ respectively. In the case $r \geqslant 2$, let L be the reversed composition $b_r, \dots, b_1$. The composition $\phi(\alpha)$ of $n+3$ is obtained from L after going through the following process:

- if b_{r-1} is even, then add 1 to b_r; otherwise, append 1 at the beginning of L;
- if b_1 is even, then add 2 to b_1; otherwise, append 2 at the end of L.

For instance, if $\alpha = UD_2UUDUD_2UDUDUUD$ then we have $B_1 = U$, $B_2 = D_2U$, $B_3 = UDU$, $B_4 = D_2UDUDU$, $B_5 = UD$, $b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 6, b_5 = 2$, $L = 2, 6, 3, 2, 1$, and $\phi(\alpha) = 3, 6, 3, 2, 1, 2$.

In the case $r = 1$, α is either $(UD)^{n/2}$ or $(DU)^{n/2}$. So, we define $\phi((UD)^{n/2}) = n+1, 2$, and $\phi((DU)^{n/2}) = 1, n+2$ (these are valid compositions since n has to be even in these cases).

In the case $r = 0$, α is empty, and we define $\phi(\varepsilon) = 1, 2$.

Due to the definition, it is clear that the composition $\phi(\alpha)$ belongs to $\mathcal{C}'(n+3)$, for any $n \geqslant 0$.

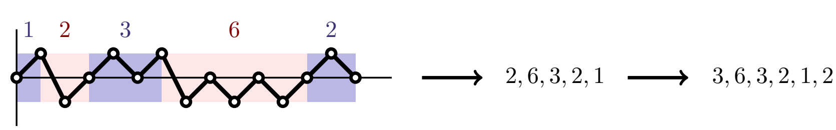

Figure 4: The image by ϕ of $\alpha = UD_2UUDUD_2UDUDUUD$ is $\phi(\alpha) = 3, 6, 3, 2, 1, 2$.

Theorem 7. *The map ϕ from $\mathcal{G}_n^{[-1,1]}$ to $\mathcal{C}'(n+3)$ is a bijection.*

Proof. Since $\mathcal{G}_n^{[-1,1]}$ and $\mathcal{C}'(n+3)$ have the same cardinality (see the end of subsection 7.1), it suffices to prove that ϕ is surjective.

Let $c = c_1, \ldots, c_r$ be an element of $\mathcal{C}'(n+3)$, with $r \geqslant 2$ (the case $r = 1$ does not occur since c_1 is odd and c_r is even implies $r \geqslant 2$). Thus, r is necessarily even. We distinguish four cases: (i) $c_1 = 1$ and $c_r = 2$, (ii) $c_1 = 1$ and c_r is even and greater than 2, (iii) c_1 is odd and greater than 1 and $c_r = 2$, (iv) c_1 is odd and greater than 1 and c_r is even and greater than 2.

According to each case, we define $\alpha \in \mathcal{G}_n^{[-1,1]}$ such that $\phi(\alpha) = c$:
Case (i): Since c_{r-1} is odd, c_{r-2} is even, and so on, and finally c_2 is even.

$$\alpha = (UD)^{(c_{r-1}-1)/2}U \; D_2U(DU)^{(c_{r-2}-2)/2} \ldots (UD)^{(c_3-1)/2}U \; D_2U(DU)^{(c_2-2)/2}.$$

Case (ii):

$$\alpha = (DU)^{(c_r-2)/2} \; (UD)^{(c_{r-1}-1)/2}U \ldots (UD)^{(c_3-1)/2}U \; D_2U(DU)^{(c_2-2)/2}.$$

Case (iii):

$$\alpha = (UD)^{(c_{r-1}-1)/2}U \; D_2U(DU)^{(c_{r-2}-2)/2} \ldots D_2U(DU)^{(c_2-2)/2} \; (UD)^{(c_1-1)/2}.$$

Case (iv):

$$\alpha = (DU)^{(c_r-2)/2} \; (UD)^{(c_{r-1}-1)/2}U \ldots D_2U(DU)^{(c_2-2)/2} \; (UD)^{(c_1-1)/2}.$$

For each case, it is clear that α belongs to $\mathcal{G}_n^{[-1,1]}$, which implies that ϕ is surjective, and then bijective. □

8 DAP with a special first return decomposition

Recently in [1], the authors introduced and enumerated the subset $\mathcal{D}^{h,\geqslant}$ of restricted Dyck paths defined as follows: the set $\mathcal{D}^{h,\geqslant}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{h,\geqslant}, \\ h(U\alpha D) \geqslant h(\beta), \end{cases} \tag{5}$$

where $h(\alpha)$ is the maximal ordinate reached by the path α. The authors prove algebraically and bijectively that n-length paths in $\mathcal{D}^{h,\geqslant}$ are in one-to-one correspondence with Motzkin paths of length n. Based on this decomposition and in the same way as for Dyck paths, we define a subset of $\mathcal{A} \cup \{\varepsilon\} \subset \mathcal{G}$ as follows. The set $\mathcal{H}$ is the union of the empty path with all DAP $\gamma \in \mathcal{A}$ having a first return decomposition satisfying the following condition:

(C) $\gamma = \alpha\beta$ with $\alpha \in \mathcal{P} \cup \{UD\}$, and $\alpha^\flat \in \mathcal{H}$ whenever $\alpha \neq UD, \beta \in \mathcal{H}$ and $h(\alpha) \geqslant h(\beta)$.

For $n \geqslant 0$, we denote by $\mathcal{H}_n$ the set of DAP of length n in $\mathcal{H}$. For instance, we have $\mathcal{H}_0 = \{\varepsilon\}$, $\mathcal{H}_1 = \varnothing$, $\mathcal{H}_2 = \{UD\}$, $\mathcal{H}_3 = \{UUD_2\}$, $\mathcal{H}_4 = \{UUUD_3, UDUD\}$, $\mathcal{H}_5 = \{UUUUD_4, UUDUD_2, UUD_2UD\}$.

In this section, we enumerate the set $\mathcal{H}_n$. For $k \geqslant 0$, let $A_k(x) = \sum_{n\geqslant 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\geqslant 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of DAP in $\mathcal{H}_n$ having a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^{k} A_i(x)$ and the generating function for the set $\mathcal{H}$, namely $B(x)$, is given by $B(x) = \lim_{k\to\infty} B_k(x)$.

Due to the definition of $\mathcal{H}$, we have

$$\begin{cases} A_0(x) &= B_0(x) = 1, \\ A_1(x) &= x^2 A_0(x) B_1(x), \\ A_k(x) &= x A_{k-1}(x) B_k(x). \end{cases}$$

Lemma 1. *For $k \geqslant 1$, we have*

$$B_{k-1}(x) = \frac{(1 - x^3 + x)B_k(x) - 1}{x^2 B_k(x) + x}.$$

Proof. We proceed by induction on k. Since $B_0(x) = 1$ and $B_1(x) = \frac{x^2}{1-x^2}$ it is easy to check that $B_0(x) = \frac{(1-x^3+x)B_1(x)-1}{x^2B_1(x)+x}$.

Now, assume that $B_{i-1}(x) = \frac{(1-x^3+x)B_i(x)-1}{x^2B_i(x)+x}$ for $1 \leqslant i \leqslant k-2$, we prove the result for $i = k-1$. From the above equations, and the recurrence hypothesis on $B_{k-2}(x)$, we obtain

$$\begin{aligned} B_k(x) &= A_k(x) + B_{k-1}(x) \\ &= xA_{k-1}(x)B_k(x) + B_{k-1}(x) \\ &= x(B_{k-1} - B_{k-2})B_k(x) + B_{k-1}(x) \\ &= x(B_{k-1} - \tfrac{(1-x^3+x)B_{k-1}(x)-1}{x^2B_{k-1}(x)+x})B_k(x) + B_{k-1}(x). \end{aligned}$$

Isolating $B_{k-1}(x)$, we obtain $B_{k-1}(x) = \frac{(1-x^3+x)B_k(x)-1}{x^2B_k(x)+x}$, which completes the induction. □

Taking the limit in the relation of Lemma 1 whenever k tends to ∞, we obtain

$$B(x) = \frac{(1+x-x^3)B(x)-1}{x^2B(x)+x},$$

which induces the following result.

Theorem 8. *The o.g.f. that counts the set $\mathcal{H}$ with respect to the length is given by*

$$B(x) = \frac{1-x^3-\sqrt{x^6-2\,x^3-4\,x^2+1}}{2x^2}.$$

The above generating function $B(x)$ counts also Motzkin paths of length n avoiding the patterns UH, HU and HH, see sequence A329699. The first terms of its series expansion are: $1+x^2+x^3+2x^4+3x^5+6x^6+10x^7+20x^8+36x^9+72x^{10}+136x^{11}+273x^{12}$.

We finish this part with a natural question.

Open question: The sets $\mathcal{H}$ and that of $\{UH, HU, HH\}$-avoiding Motzkin paths are thus in bijection and it would be interesting to exhibit a constructive bijection between them.

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