Counting Subwords in Hexagonal Circle Packing Paths

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Abstract

We study the distribution of a prescribed subword of length at most two in packing paths and their partial counterparts. Using a unified approach based on trivariate generating functions, we derive exact formulas for the number of such paths according to their width, final height, and number of occurrences of the subword. We also determine the total number of occurrences of each subword among all (partial) packing paths of a given width.

As a consequence, we obtain complete classifications of subwords of length at most two with respect to both avoidance and total occurrence statistics. In addition, we establish several bijections between packing paths of width $4n$ and classical combinatorial structures, including binary words, colored Dyck paths, and Schröder paths. These bijections provide new combinatorial interpretations for several enumerative sequences arising from subword avoidance in packing paths.

1 Introduction

A *hexagonal circle packing* is an arrangement of congruent circles in the plane whose centers form a triangular lattice. Neighboring circles are tangent to one another, and each circle has six tangent neighbors. The left-hand side of Figure 1 shows this packing. The tangency relations between neighboring circles determine a natural family of directed paths. We place the origin at the tangency point of two horizontally aligned circles and assume that each circle has radius 2.

The paths considered in this paper follow arcs of the packing in three possible directions: northeast, east, and southeast. Each direction may be followed along either a convex arc or a concave arc. Thus there are six elementary steps. Let $\mathcal{S} = \{U, F, D, \overline{U}, \overline{F}, \overline{D}\}$ denote the set of these steps. We illustrate the six arcs as follows:

$U =$, $F =$, $D =$, $\overline{U} =$, $\overline{F} =$, $\overline{D} =$.

A *partial packing path* is a concatenation of steps from S that starts at the origin, remains in the first quadrant, and follows the geometry of the hexagonal circle packing. A *packing path* is a partial packing path whose endpoint lies on the x-axis. The right-hand side of Figure 1 gives an example of a partial packing path consisting of the 14 steps U, $\overline{U}$, U, F, $\overline{F}$, F, D, U, F, $\overline{F}$, $\overline{U}$, $\overline{D}$, $\overline{F}$, $\overline{U}$ in this order.

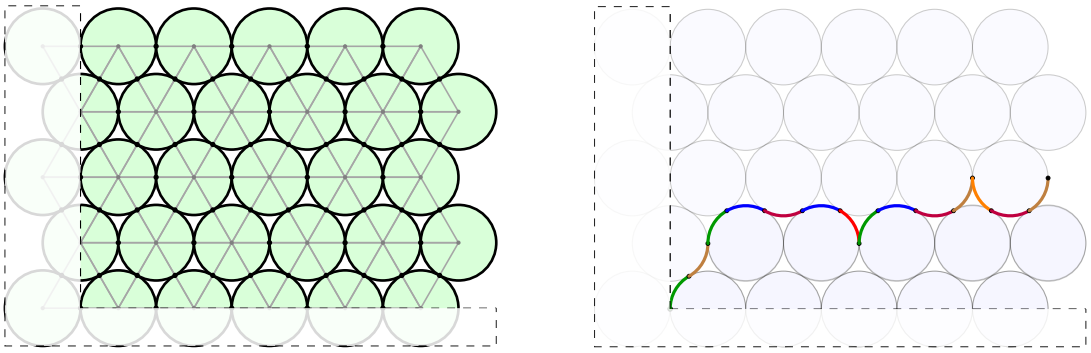

Figure 1: The hexagonal circle packing and a partial packing path.

Baril and Ramírez [3] introduced and studied partial packing paths in the hexagonal circle packing. They enumerated these paths with respect to several natural parameters, including width, height, and area, and observed that the study of subwords provides a natural continuation of their work. The present paper develops this direction by studying the number of occurrences of a prescribed subword in partial packing paths.

Let $\mathcal{P}$ denote the set of all partial packing paths, and let $\mathcal{Q}$ denote the set of all packing paths. The *width* of a path $P \in \mathcal{P}$, denoted by $\texttt{width}(P)$, is the abscissa of its endpoint; that is, the total horizontal displacement accumulated along P. Each diagonal step of type U, D, $\overline{U}$, or $\overline{D}$ increases the width by one, whereas each horizontal step of type F or $\overline{F}$ increases the width by two. We let ε denote the empty path, which has width 0. For $n \geq 0$, let $\mathcal{P}_n$ and $\mathcal{Q}_n$ denote the sets of paths of width n in $\mathcal{P}$ and $\mathcal{Q}$, respectively.

The *final height*, or simply the *height*, of a path P, denoted by $\texttt{height}(P)$, is the integer such that the ordinate of the endpoint of P is $\texttt{height}(P)\sqrt{3}$. Each step U or $\overline{U}$ increases the height by one, each step D or $\overline{D}$ decreases the height by one, and each horizontal step F or $\overline{F}$ leaves the height unchanged.

For example, the path on the right-hand side of Figure 1 is the partial packing path $P = U\overline{U}UF\overline{F}FDUF\overline{FUDFU}$. It satisfies $\mathtt{width}(P) = 20$ and $\mathtt{height}(P) = 4$.

The sequence that counts packing paths of width $4n$ is the OEIS sequence A086871 [12], which also counts skew Dyck paths of semilength n with down-steps in two colors.

We identify a partial packing path with the word over $\mathcal{S}$ determined by its sequence of steps. A word w over $\mathcal{S}$ occurs as a *subword* of a partial packing path P if there exist possibly empty words u_1 and u_2 over $\mathcal{S}$ such that $P = u_1 w u_2$. Equivalently, w occurs in P as a consecutive block of steps.

The enumeration of occurrences of prescribed subwords is a classical problem in enumerative combinatorics. For instance, Deutsch [4] studied the distribution of the subwords of length 2 in Dyck words. Subwords of length 3 in Dyck paths and in non-decreasing Dyck paths have also been studied by Deutsch [4], Flórez et al. [5], Mansour [6], Merlini et al. [8], Sapounakis et al. [13], and Sun [14].

The present paper focuses on enumerating occurrences of subwords of lengths 1 and 2 in partial packing paths. We express the corresponding counting sequences in terms of generating functions. Several of these sequences already appear in the OEIS [12], often with different combinatorial interpretations. For some of these coincidences, we provide explicit bijections. For other sequences, our results seem to give the first combinatorial interpretation recorded in the OEIS. We also obtain several sequences that appear to be new.

For a word w over $\mathcal{S}$, let $|P|_w$ denote the number of occurrences of w as a subword of P, including overlapping occurrences. We define

$$S_w(x, u, z) = \sum_{P \in \mathcal{P}} x^{\mathtt{width}(P)} u^{\mathtt{height}(P)} z^{|P|_w}.$$

Thus the coefficient of $x^n u^k z^\ell$ in $S_w(x, u, z)$ is the number of partial packing paths of width n and height k with exactly ℓ occurrences of w as a subword. We denote this coefficient by $s_w(n, k, \ell)$.

Several specializations of $S_w(x, u, z)$ are of particular interest. We set $S_w(x, u) := S_w(x, u, 0)$ and define $s_w(n, k) = [x^n u^k] S_w(x, u)$. Since setting $z = 0$ selects paths with no occurrence of w, the number $s_w(n, k)$ counts partial packing paths of width n and height k that avoid the subword w. We also define $q_w(n, \ell) = [x^n z^\ell] S_w(x, 1, z)$. Thus $q_w(n, \ell)$ counts partial packing paths of width n with exactly ℓ occurrences of w, summed over all possible final heights.

We also write

$$s_w(n) = \sum_{k \geq 0} s_w(n, k)$$

for the total number of partial packing paths of width n that avoid w. Since packing paths are precisely the partial packing paths of height 0, we write $c_w(n) = s_w(n, 0)$ for the number of packing paths of width n that avoid w.

Let $g_w(n)$ denote the total number of occurrences of w among all paths in $\mathcal{P}_n$. Its generating function is obtained by differentiating $S_w(x, 1, z)$ with respect to z and then

setting $z = 1$:

$$\sum_{n\geq 0} g_w(n)x^n = \left.\frac{\partial S_w(x,1,z)}{\partial z}\right|_{z=1}.$$

Similarly, let $d_w(n)$ denote the total number of occurrences of w among all paths in $\mathcal{Q}_n$, that is, among all packing paths of width n. Then

$$\sum_{n\geq 0} d_w(n)x^n = \left.\frac{\partial S_w(x,0,z)}{\partial z}\right|_{z=1}.$$

Outline of the paper. In this paper, we present a study of the enumeration of partial packing paths and packing paths according to the number of occurrences of a given subword of length at most two. In Section 2, we first describe the general method used to derive trivariate generating functions and exact formulas for the number of such paths with respect to their width, final height, and number of occurrences of the subword. We also explain how to compute the total number of occurrences of a given subword among all partial packing paths (respectively, packing paths).

In Section 2.1, we illustrate the method in the case of the subword $\overline{U}\overline{D}$, providing complete proofs. We then present (see Sections 2.2 and 2.3) the trivariate generating functions for all subwords of length at most two, omitting the details of the proofs since the same method applies throughout.

In Section 3, we establish bijections between packing paths of width $4n$ and certain sets (or subsets) of classical combinatorial objects, including binary words, colored Dyck paths, and Schröder paths.

Finally, in Section 4, we summarize our results in four tables. The first two tables classify subwords of length at most two according to the number of partial packing paths and packing paths of width n that avoid them. The last two tables classify these subwords according to the total number of their occurrences among all partial packing paths and packing paths of width n.

2 A generating function approach

We first recall the generating function approach for packing paths counted with respect to width and final height. Baril and Ramírez obtained this enumeration in an earlier paper [3], but we include the setup here because it is the starting point for the refined generating functions $S_w(x,u,z)$ when w has length 1 or 2.

The admissible subwords of length 1 are the six elements of $\mathcal{S} = \{U, F, D, \overline{U}, \overline{F}, \overline{D}\}$. The local geometry of the packing allows only twelve admissible subwords of length 2, namely

$$\begin{array}{cccccc} U\overline{U}, & UF, & \overline{U}U, & \overline{U}\overline{D}, & FD, & F\overline{F}, \\ \overline{F}U, & \overline{F}F, & DU, & D\overline{D}, & \overline{D}D, & \overline{D}\overline{F}. \end{array}$$

These are the subwords for which we develop the refined enumeration below.

For $k \geq 0$, let $f_k = f_k(x)$, $g_k = g_k(x)$, and $h_k = h_k(x)$ be the generating functions in which the coefficient of x^n counts partial packing paths of width n and height k whose last step is, respectively, of type U or $\overline{U}$, of type D or $\overline{D}$, and of type F or $\overline{F}$.

We further split these generating functions according to whether the last step is unbarred or barred. Thus f_k^0, g_k^0, and h_k^0 count paths ending with U, D, and F, respectively, while f_k^1, g_k^1, and h_k^1 count paths ending with $\overline{U}$, $\overline{D}$, and $\overline{F}$, respectively. Hence, for all $k \geq 0$,

$$f_k = f_k^0 + f_k^1, \qquad g_k = g_k^0 + g_k^1, \qquad h_k = h_k^0 + h_k^1.$$

For $i \in \{0, 1\}$, we define

$$F^i(u) = \sum_{k \geq 0} f_k^i u^k, \qquad G^i(u) = \sum_{k \geq 0} g_k^i u^k, \qquad H^i(u) = \sum_{k \geq 0} h_k^i u^k.$$

We also set

$$S(x, u) = F^0(u) + F^1(u) + G^0(u) + G^1(u) + H^0(u) + H^1(u).$$

By convention, the empty path contributes to $f_0(x)$ and $f_0^0(x)$. Thus $F^0(u) - 1$ is the generating function for the nonempty partial packing paths ending with U.

The decomposition according to the last step gives the following system of equations:

$$\begin{aligned}
F^0(u) &= 1 + xu(1 + F^1(u) + G^0(u)),\\
F^1(u) &= xu(F^0(u) - 1 + H^1(u)),\\
G^0(u) &= \frac{x}{u}(G^1(u) + H^0(u)),\\
G^1(u) &= \frac{x}{u}(F^1(u) + G^0(u) - g_0^0),\\
H^0(u) &= x^2(F^0(u) - 1 + H^1(u)),\\
H^1(u) &= x^2(G^1(u) + H^0(u)).
\end{aligned} \tag{1}$$

Baril and Ramírez solved this system in [3]. Here we use (1) as the basic system from which the refinements involving occurrences of subwords of lengths 1 and 2 are obtained.

The local restrictions on consecutive steps can be encoded by a transfer matrix. We order the six components as

$$F^0,\ F^1,\ G^0,\ G^1,\ H^0,\ H^1,$$

corresponding, respectively, to paths ending with

$$U,\ \overline{U},\ D,\ \overline{D},\ F,\ \overline{F}.$$

Let

$$\boldsymbol{X}(u) = \begin{pmatrix} F^0(u) \\ F^1(u) \\ G^0(u) \\ G^1(u) \\ H^0(u) \\ H^1(u) \end{pmatrix}.$$

Then (1) can be written as

$$\boldsymbol{X}(u) = \boldsymbol{b}(u) + M(u)\boldsymbol{X}(u) - \boldsymbol{q}(u)g_0^0,$$

where

$$\boldsymbol{b}(u) = \begin{pmatrix} 1+xu \\ -xu \\ 0 \\ 0 \\ -x^2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{q}(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x/u \\ 0 \\ 0 \end{pmatrix},$$

and

$$M(u) = \begin{pmatrix} 0 & xu & xu & 0 & 0 & 0 \\ xu & 0 & 0 & 0 & 0 & xu \\ 0 & 0 & 0 & x/u & x/u & 0 \\ 0 & x/u & x/u & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & x^2 & x^2 & 0 \end{pmatrix}.$$

With this ordering, the entry in row i and column j gives the contribution of appending the step corresponding to row i to a path whose last step corresponds to column j. Thus the nonzero entries of $M(u)$ are precisely the admissible consecutive subwords of length 2.

Let $w = ab$ be an admissible subword of length 2. Suppose that the step a corresponds to column j and that the step b corresponds to row i. Let E_{ij} denote the matrix with a 1 in position (i, j) and zeros elsewhere, and let $m_{ij}(u)$ be the corresponding nonzero entry of $M(u)$. To mark each occurrence of w by the variable z, we replace this entry by $zm_{ij}(u)$. Equivalently, we define

$$M_w(u, z) = M(u) + (z-1)m_{ij}(u)E_{ij}.$$

All other entries remain unchanged.

The generating functions for partial packing paths in which occurrences of w are marked by z are therefore determined by the following system, subject to the adjustments below

$$\boldsymbol{X}_w(u, z) = \boldsymbol{b}(u) + M_w(u, z)\boldsymbol{X}_w(u, z) - \boldsymbol{q}(u)g_{w,0}^0,$$

where $g_{w,0}^0 = [u^0]G_w^0(u, z)$.

For $w = U\overline{U}$, replace $-xu$ by $-zxu$ in $\boldsymbol{b}(u)$. For $w = UF$, replace $-x^2$ by $-zx^2$ in $\boldsymbol{b}(u)$. These two adjustments ensure that the correction for the empty path in $F_w^0(u, z)$ carries the same weight as the marked transition. For $w = D\overline{D}$, replace $\boldsymbol{q}(u)$ by $z\boldsymbol{q}(u)$, since the forbidden transition from D to $\overline{D}$ at height zero is also marked by z.

After solving this linear system in terms of the boundary term $g_{w,0}^0$, we determine $g_{w,0}^0$ by the kernel method [1,2,9–11]. More precisely, the components of $\boldsymbol{X}_w(u, z)$ are rational functions whose denominators contain a common kernel. Since these components are generating

functions for paths that remain in the first quadrant, they must be formal power series in u with non-negative powers only. We therefore cancel the numerator at the small zero of the kernel. This condition determines the boundary term $g^0_{w,0}$.

Finally, the corresponding trivariate generating function is

$$S_w(x,u,z) = F^0_w(u,z) + F^1_w(u,z) + G^0_w(u,z) + G^1_w(u,z) + H^0_w(u,z) + H^1_w(u,z).$$

2.1 The subword $\overline{UD}$

We illustrate the method with the subword $w = \overline{UD}$. The steps $\overline{U}$ and $\overline{D}$ correspond to the components F^1 and G^1, respectively. Hence this subword is represented in the transfer matrix by the entry in row G^1 and column F^1. This entry is x/u. To record occurrences of $\overline{UD}$, we replace this entry by zx/u and leave all other entries unchanged. Thus, whenever a partial packing path is extended from a last step $\overline{U}$ to a last step $\overline{D}$, the corresponding contribution is multiplied by z.

Theorem 1. *Let*

$$\Delta = \Delta(x,z) = 1 - 2(3+2z)x^4 + (1+2z)^2x^8.$$

Then

$$S_{\overline{UD}}(x,u,z) = \frac{x\big(3-(1+2z)x^4+\sqrt{\Delta}\big) + u\big(1+4x^2+(1+2z)x^4-\sqrt{\Delta}\big)}{2x\big(1-(1+2z)x^4+\sqrt{\Delta}-2u^2x^2\big)}.$$

Proof. As explained above, the subword $\overline{UD}$ is recorded by replacing the entry in row G^1 and column F^1 of the transfer matrix by zx/u. Thus the transfer matrix is

$$M_{\overline{UD}}(u,z) = \begin{pmatrix} 0 & xu & xu & 0 & 0 & 0 \\ xu & 0 & 0 & 0 & 0 & xu \\ 0 & 0 & 0 & x/u & x/u & 0 \\ 0 & zx/u & x/u & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & x^2 & x^2 & 0 \end{pmatrix}.$$

Equivalently, the only modified equation is the equation for $G^1(u)$:

$$G^1(u) = \frac{x}{u}\big(zF^1(u) + G^0(u) - g^0_0\big),$$

where $g_0^0 = G^0(0)$. All other equations are unchanged. Hence

$$\begin{aligned}
F^0(u) &= 1 + xu\big(1 + F^1(u) + G^0(u)\big),\\
F^1(u) &= xu\big(F^0(u) - 1 + H^1(u)\big),\\
G^0(u) &= \frac{x}{u}\big(G^1(u) + H^0(u)\big),\\
G^1(u) &= \frac{x}{u}\big(zF^1(u) + G^0(u) - g_0^0\big),\\
H^0(u) &= x^2\big(F^0(u) - 1 + H^1(u)\big),\\
H^1(u) &= x^2\big(G^1(u) + H^0(u)\big).
\end{aligned} \tag{2}$$

We determine g_0^0 by applying the kernel method to $F^1(u)$. Solving the linear system (2) gives

$$F^1(u) = \frac{u^2x^2\big(2x^2g_0^0 - u^2 + x^2\big)}{u^4x^2 + (1+2z)u^2x^4 - u^2 + x^2}.$$

The kernel is

$$K(u) = u^4x^2 + (1+2z)u^2x^4 - u^2 + x^2.$$

Writing

$$K(u) = x^2(u^2 - r)(u^2 - s),$$

we have

$$r = \frac{1 - (1+2z)x^4 - \sqrt{\Delta}}{2x^2}, \qquad s = \frac{1 - (1+2z)x^4 + \sqrt{\Delta}}{2x^2},$$

where

$$\Delta = 1 - 2(3+2z)x^4 + (1+2z)^2x^8.$$

The small solution is r. Hence the numerator of $F^1(u)$ must vanish at $u^2 = r$. This gives $2x^2g_0^0 - r + x^2 = 0$, and therefore

$$g_0^0 = \frac{r - x^2}{2x^2}.$$

Substituting this value of g_0^0 into the six generating functions and summing them, we obtain

$$S_{\overline{UD}}(x,u,z) = F^0(u) + F^1(u) + G^0(u) + G^1(u) + H^0(u) + H^1(u).$$

After cancellation of the kernel factor $u^2 - r$, this expression simplifies to

$$S_{\overline{UD}}(x,u,z) = \frac{u\big(rx^2 + (1+2z)x^4 + 2x^2\big) - rx^3 - (1+2z)x^5 + 2x}{2x^3(s - u^2)}.$$

Finally, replacing r and s by their explicit expressions gives the stated formula. □

The first terms in the series expansion of $S_{\overline{UD}}(x,u,z)$ are

$$\begin{aligned}1 &+ ux + u^2x^2 + (u + u^3 + uz)x^3 + (1 + u^4 + z)x^4 \\ &+ (2u + u^3 + u^5 + 2uz + u^3z)x^5 + (3u^2 + u^6 + 3u^2z)x^6 \\ &+ (3u + 4u^3 + u^5 + u^7 + 5uz + 4u^3z + u^5z + 2uz^2)x^7 \\ &+ (3 + 5u^4 + u^8 + 5z + 5u^4z + 2z^2)x^8 \\ &+ (6u + 5u^3 + 6u^5 + u^7 + u^9 + 10uz + 9u^3z + 6u^5z + u^7z + 4uz^2 + \mathbf{4u^3z^2})x^9 + O(x^{10}).\end{aligned}$$

The coefficient of $x^9u^3z^2$ is 4. Hence there are exactly four partial packing paths of width 9 and final height 3 having exactly two occurrences of the subword $\overline{UD}$. These four paths are shown in Figure 2.

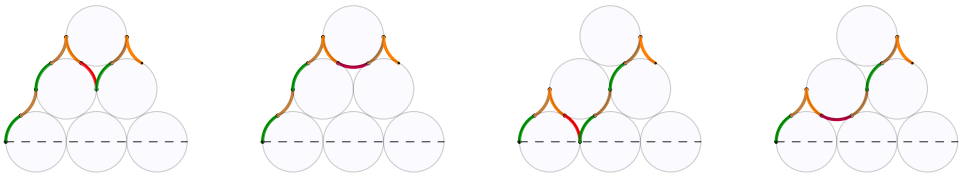

Figure 2: The four partial packing paths of width 9 and final height 3 having exactly two occurrences of the subword $\overline{UD}$.

Using the generating function in Theorem 1 together with Lagrange inversion [7], we obtain explicit formulas for the coefficients $s_{\overline{UD}}(n,k,\ell)$.

Theorem 2. *Let*

$$S_{\overline{UD}}(x,u,z) = \sum_{n,k,\ell\geq 0} s_{\overline{UD}}(n,k,\ell)x^n u^k z^\ell.$$

Then, for $n,k,\ell \geq 0$, we have

$$\begin{aligned}s_{\overline{UD}}(4n+2k,2k,\ell) &= \frac{1}{2}\big(T(n,k+1,\ell) + T(n,k,\ell)\big),\\ s_{\overline{UD}}(4n+2k+1,2k+1,\ell) &= T(n,k+1,\ell),\\ s_{\overline{UD}}(4n+2k+3,2k+1,\ell) &= \frac{1}{2}\big(T(n+1,k+1,\ell) - T(n+1,k,\ell)\big).\end{aligned}$$

All coefficients not covered by these three cases are zero. Here $T(n,0,\ell) = \delta_{n,0}\cdot\delta_{\ell,0}$ and $T(0,m,\ell) = \delta_{\ell,0}$, and, for $n \geq 1$ and $m \geq 1$,

$$T(n,m,\ell) = \frac{m}{n}\sum_{b=1}^{n} 2^b \binom{n}{b}\binom{b}{\ell}\binom{m+n-1}{b-1}.$$

Proof. Let $q = x^4$ and define $R = R(q,z)$ by

$$R = \frac{1-(1+2z)q - \sqrt{1-2(3+2z)q+(1+2z)^2q^2}}{2q} = \frac{1-(1+2z)q-\sqrt{\Delta}}{2q}.$$

Then R is the unique formal power series in q, with constant term 1, satisfying

$$R = 1 + qR(R + 1 + 2z).$$

Moreover,

$$\sqrt{\Delta} = 1 - (1 + 2z)x^4 - 2x^4R.$$

Substituting this expression into the formula given in Theorem 1 gives

$$S_{\overline{UD}}(x, u, z) = \frac{x(R + 1) + u\big(R(1 + 2x^2) - 1\big)}{2x(1 - u^2x^2R)}.$$

Expanding the denominator as a geometric series, we obtain

$$S_{\overline{UD}}(x, u, z) = \frac{1}{2}\sum_{j\geq 0} u^{2j}x^{2j}\left(R^{j+1} + R^j\right) + \frac{1}{2}\sum_{j\geq 0} u^{2j+1}x^{2j-1}\left(R^{j+1} - R^j + 2x^2R^{j+1}\right).$$

Therefore,

$$[u^{2k}]S_{\overline{UD}}(x, u, z) = \frac{1}{2}x^{2k}\left(R^{k+1} + R^k\right) \tag{3}$$

and

$$[u^{2k+1}]S_{\overline{UD}}(x, u, z) = \frac{1}{2}x^{2k-1}\left(R^{k+1} - R^k\right) + x^{2k+1}R^{k+1}. \tag{4}$$

We now compute the coefficients of powers of R. Put $T(n, m, \ell) = [q^n z^\ell]R(q, z)^m$. For $m = 0$, since $R^0 = 1$, we have $T(n, 0, \ell) = \delta_{n,0} \cdot \delta_{\ell,0}$. For $m \geq 1$, write $W = R - 1$. Then

$$W = q(1 + W)(W + 2 + 2z).$$

By Lagrange inversion, for $n \geq 1$ and $m \geq 1$,

$$[q^n z^\ell]R^m = [q^n z^\ell](1 + W)^m = \frac{m}{n}[y^{n-1}z^\ell](1 + y)^{m-1}\left((1 + y)(y + 2 + 2z)\right)^n.$$

Hence

$$\begin{aligned} T(n, m, \ell) = [q^n z^\ell]R^m =& \frac{m}{n}[y^{n-1}z^\ell](1 + y)^{m+n-1}(y + 2 + 2z)^n \\ =& \frac{m}{n}\sum_{b=1}^{n}\binom{n}{b}\binom{m + n - 1}{b - 1}[z^\ell](2 + 2z)^b \\ =& \frac{m}{n}\sum_{b=1}^{n} 2^b\binom{n}{b}\binom{b}{\ell}\binom{m + n - 1}{b - 1}. \end{aligned}$$

Together with the initial value $T(0, m, \ell) = \delta_{\ell,0}$, this gives the stated expression for $T(n, m, \ell)$.

It remains to extract powers of x. Since $q = x^4$, we have

$$[x^a z^\ell] x^c R(q,z)^m = \begin{cases} T\left(\dfrac{a-c}{4}, m, \ell\right), & a \geq c \text{ and } a \equiv c \pmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

Equation (3) gives

$$s_{\overline{UD}}(4n+2k, 2k, \ell) = \frac{1}{2}\left(T(n, k+1, \ell) + T(n, k, \ell)\right).$$

For odd height, we use (4). The term $x^{2k+1}R^{k+1}$ contributes to $x^{4n+2k+1}$, and therefore

$$s_{\overline{UD}}(4n+2k+1, 2k+1, \ell) = T(n, k+1, \ell).$$

The term $\frac{1}{2}x^{2k-1}\left(R^{k+1} - R^k\right)$ contributes to $x^{4n+2k+3}$, since $4n+2k+3 = 2k-1+4(n+1)$. Thus

$$s_{\overline{UD}}(4n+2k+3, 2k+1, \ell) = \frac{1}{2}\left(T(n+1, k+1, \ell) - T(n+1, k, \ell)\right).$$

All remaining congruence classes give zero coefficients. This completes the proof. □

Theorem 1 gives the following specializations. First, let

$$S_{\overline{UD}}(x,u) := S_{\overline{UD}}(x,u,0) = \sum_{n,k\geq 0} s_{\overline{UD}}(n,k) x^n u^k$$

be the generating function for partial packing paths avoiding the consecutive subword $\overline{UD}$, where x marks the width and u marks the final height. Then

$$S_{\overline{UD}}(x,u) = \frac{x(3-x^4) + u(1+4x^2+x^4) + (x-u)\sqrt{1-6x^4+x^8}}{2x\left(1-2u^2x^2-x^4+\sqrt{1-6x^4+x^8}\right)}.$$

Setting $u = 1$, we obtain

$$S_{\overline{UD}}(x,1) = \sum_{n\geq 0} s_{\overline{UD}}(n) x^n = \frac{1+3x+4x^2+x^4-x^5-(1-x)\sqrt{1-6x^4+x^8}}{2x\left(1-x^2(2+x^2)+\sqrt{1-6x^4+x^8}\right)}.$$

The first few values of $s_{\overline{UD}}(n)$ are

$$1,\ 1,\ 1,\ 2,\ 2,\ 4,\ 4,\ 9,\ 9,\ 19,\ 19,\ 44,\ 44,\ 96,\ 96,\ 225,\ 225,\ 501,\ 501, \ldots.$$

This sequence is obtained from A247623 by repeating each term twice except the initial term, which occurs once.

Setting $u = 0$, we obtain the generating function for packing paths avoiding $\overline{UD}$:

$$S_{\overline{UD}}(x,0) = \sum_{n\geq 0} c_{\overline{UD}}(n) x^n = \frac{1+x^4-\sqrt{1-6x^4+x^8}}{4x^4}.$$

In particular, $c_{\overline{UD}}(4n) = s_n$, where s_n is the n-th small Schröder number A001003, and $c_{\overline{UD}}(m) = 0$ otherwise. The first few values of $c_{\overline{UD}}(4n)$ are

$$1,\ 1,\ 3,\ 11,\ 45,\ 197,\ 903,\ 4279,\ 20793,\ 103049,\ 518859,\ 2646723,\ 13648869, \ldots.$$

We also obtain generating functions for the total number of occurrences of $\overline{UD}$. For partial packing paths, summed over all final heights, we have

$$\sum_{n\geq 0} g_{\overline{UD}}(n)x^n = \left.\frac{\partial S_{\overline{UD}}(x,1,z)}{\partial z}\right|_{z=1} = \frac{x^3(1+x)}{(1-3x^2)\sqrt{1-10x^4+9x^8}}.$$

The first few values of $g_{\overline{UD}}(n)$ are

$$0,\ 0,\ 0,\ 1,\ 1,\ 3,\ 3,\ 14,\ 14,\ 42,\ 42,\ 159,\ 159,\ 477,\ 477,\ 1676, \ldots.$$

The subsequence $g_{\overline{UD}}(4n)$ coincides with A383946. Our interpretation in terms of occurrences of $\overline{UD}$ in partial packing paths appears to be the first combinatorial interpretation recorded for this sequence.

For packing paths, we have

$$\sum_{n\geq 0} d_{\overline{UD}}(n)x^n = \left.\frac{\partial S_{\overline{UD}}(x,0,z)}{\partial z}\right|_{z=1} = \frac{1-3x^4-\sqrt{1-10x^4+9x^8}}{2\sqrt{1-10x^4+9x^8}}.$$

The first few values of $d_{\overline{UD}}(4n)$ are

$$0,\ 1,\ 9,\ 73,\ 593,\ 4881,\ 40665,\ 342105,\ 2900385,\ 24742945,\ 212155049, \ldots.$$

This sequence is A388060. Again, our interpretation in terms of occurrences of the subword $\overline{UD}$ in packing paths appears to be the first combinatorial interpretation recorded for this sequence.

In the next subsection, we present the generating functions $S_w(x,u,z)$ for all admissible subwords w of lengths 1 and 2. We omit the proofs, since they follow the same argument used for $\overline{UD}$. We obtain the explicit forms using symbolic computation software.

We also record the corresponding sequences in Tables 1, 2, 3, and 4. These tables include, among others, the sequences $s_w(n)$, $c_w(n)$, $d_w(n)$, and $g_w(n)$ arising from the specializations described above.

2.2 The enumeration of the subwords of length 1

Here, we give the trivariate generating functions and the corresponding initial series coefficients for all subwords of length one.

- The subword U: Let

$$\Delta = \Delta(x,z) = 1 - (4+6z)x^4 + (2+z)^2x^8.$$

The generating function for partial packing paths in which occurrences of U are marked by z is

$$\begin{aligned} S_U(x,u,z) &= \frac{p(x,u,z) + (x-uz)\sqrt{\Delta}}{x(1+z)(1-2u^2x^2z - x^4(2+z) + \sqrt{\Delta})} \\ &= 1 + uzx + u^2zx^2 + (2uz+u^3z^2)x^3 + (2z+u^4z^2)x^4 \\ &\qquad + (2uz + 2uz^2 + 2u^3z^2 + u^5z^3)x^5 + O(x^6), \end{aligned}$$

where

$$p(x,u,z) = uz + 2ux^2z(1+z) - x^5(2+z) + ux^4z(2+z) + x(1+2z).$$

- The subword $\overline{U}$: Let

$$\Delta = \Delta(x,z) = 1 - 2(2+3z)x^4 + (2+z)^2x^8.$$

The generating function for partial packing paths in which occurrences of $\overline{U}$ are marked by z is

$$\begin{aligned} S_{\overline{U}}(x,u,z) &= \frac{x\big(3-(2+z)x^4\big) + u\big(1+4x^2+(2+z)x^4\big) + (x-u)\sqrt{\Delta}}{2x\big(1-2u^2zx^2 - (2+z)x^4 + \sqrt{\Delta}\big)} \\ &= 1 + ux + u^2zx^2 + (u+uz+u^3z)x^3 + (1+z+u^4z^2)x^4 \\ &\quad + (2u + 2uz + u^3z + u^3z^2 + u^5z^2)x^5 + O(x^6). \end{aligned}$$

- The subword F: Let

$$\Delta = \Delta(x,z) = 1 - 2(3+2z)x^4 + (1+2z)^2x^8.$$

The generating function for partial packing paths in which occurrences of F are marked by z is

$$\begin{aligned} S_F(x,u,z) &= \frac{x(3-x^4(1+2z)) + u(1+4x^2+x^4(1+2z)) + (x-u)\sqrt{\Delta}}{2x(1-2u^2x^2 - x^4 - 2x^4z + \sqrt{\Delta})} \\ &= 1 + ux + u^2x^2 + (u+u^3+uz)x^3 + (1+u^4+z)x^4 \\ &\qquad + (2u + u^3 + u^5 + 2uz + u^3z)x^5 + O(x^6). \end{aligned}$$

- The subword $\overline{F}$: Let

$$\Delta = \Delta(x,z) = 1 - 2(3+2z)x^4 + (1+2z)^2x^8.$$

The generating function for partial packing paths in which occurrences of $\overline{F}$ are marked by z is

$$\begin{aligned} S_{\overline{F}}(x,u,z) &= \frac{u(1+x^2)(1+(1+2z)x^2)+x(2+z-z(1+2z)x^4)+(xz-u)\sqrt{\Delta}}{x(1+z)(1-2u^2x^2-(1+2z)x^4+\sqrt{\Delta})} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(2+u^4)x^4 \\ &\qquad +(2u+2u^3+u^5+2uz)x^5+O(x^6). \end{aligned}$$

- The subword D: Let

$$\Delta = \Delta(x,z) = 1-2(2+3z)x^4+(2+z)^2x^8.$$

The generating function for partial packing paths in which occurrences of D are marked by z is

$$\begin{aligned} S_{D}(x,u,z) &= \frac{x(1+2z-x^4(2+z))+u(1+2x^2(1+z)+x^4(2+z))+(x-u)\sqrt{\Delta}}{x(1+z)(1-2u^2x^2-x^4(2+z)+\sqrt{\Delta})} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(u^4+2z)x^4 \\ &\qquad +(2u+2u^3+u^5+2uz)x^5+O(x^6). \end{aligned}$$

- The subword $\overline{D}$: Let

$$\Delta = \Delta(x,z) = 1-2(2+3z)x^4+(2+z)^2x^8.$$

The generating function for partial packing paths in which occurrences of $\overline{D}$ are marked by z is

$$\begin{aligned} S_{\overline{D}}(x,u,z) &= \frac{x(3-x^4(2+z))+u(1+4x^2+x^4(2+z))+(x-u)\sqrt{\Delta}}{2x(1-2u^2x^2-x^4(2+z)+\sqrt{\Delta})} \\ &= 1+ux+u^2x^2+(u+u^3+uz)x^3+(1+u^4+z)x^4 \\ &\qquad +(2u+u^3+u^5+2uz+u^3z)x^5+O(x^6). \end{aligned}$$

2.3 The enumeration of the subwords of length 2

Here, we give the trivariate generating functions and the corresponding initial series coefficients for all subwords of length two.

- The subword $U\overline{U}$: Let

$$\Delta = \Delta(x,z) = 1-(6+4z)x^4+(5+4z)x^8.$$

The generating function for partial packing paths in which occurrences of $U\overline{U}$ are marked by z is

$$\begin{aligned} S_{U\overline{U}}(x,u,z) &= \frac{p(x,u,z)+\big(x+ux^2-uz-ux^2z\big)\sqrt{\Delta}}{x(1+z)\big(1-3x^4-2u^2x^6(1-z)-2u^2x^2z+\sqrt{\Delta}\big)} \\ &= 1+ux+u^2zx^2+(u+uz+u^3z)x^3+(1+z+u^4z^2)x^4 \\ &\qquad +(2u+2uz+u^3z+u^3z^2+u^5z^2)x^5+O(x^6), \end{aligned}$$

where

$$p(x,u,z)=x+ux^2+2ux^4-x^5+ux^6+uz+2xz+3ux^2z+ux^4z-2x^5z-ux^6z.$$

- The subword UF: Let

$$\Delta=\Delta(x,z)=1-2x^4(4+z)+x^8(8+z^2).$$

The generating function for partial packing paths in which occurrences of UF are marked by z is

$$\begin{aligned} S_{UF}(x,u,z) &= \frac{x\big(3-x^4(4-z)+\sqrt{\Delta}\big)+u\big(1+4x^2-4x^6(1-z)+3x^4z-\sqrt{\Delta}\big)}{2x\big(1-2u^2(x^2+x^6(-1+z))-x^4(2+z)+\sqrt{\Delta}\big)} \\ &= 1+ux+u^2x^2+(u+u^3+uz)x^3+(1+u^4+z)x^4 \\ &\qquad +(2u+u^3+u^5+2uz+u^3z)x^5+O(x^6). \end{aligned}$$

The sequence $c_{UF}(4n)$ coincides with A068764 and our combinatorial interpretation is probably the first for that sequence.

- The subword $\overline{U}U$: Let

$$\Delta=\Delta(x,z)=1-2x^4(4+z)+x^8(-4+z)^2.$$

The generating function for partial packing paths in which occurrences of $\overline{U}U$ are marked by z is

$$\begin{aligned} S_{\overline{U}U}(x,u,z) &= \frac{3x+u(1+4x^2)+4u^2x^3(1-z)+(x-u)\big(\sqrt{\Delta}-(4-z)x^4\big)}{2x\big(1-(4-z)x^4-2u^2x^2z+\sqrt{\Delta}\big)} \\ &= 1+ux+u^2x^2+(2u+u^3z)x^3+(2+u^4z)x^4 \\ &\qquad +(4u+2u^3z+u^5z^2)x^5+O(x^6). \end{aligned}$$

- The subword FD: Let

$$\Delta=\Delta(x,z)=1-2x^4(4+z)+x^8(8+z^2).$$

The generating function for partial packing paths in which occurrences of FD are marked by z is

$$\begin{aligned} S_{FD}(x,u,z) &= \frac{x(3+x^4(-4+z)+\sqrt{\Delta})+u\big(1+4x^2+x^4(4-z)-\sqrt{\Delta}\big)}{2x\big(1-2u^2x^2-x^4(2+z)+\sqrt{\Delta}\big)} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(1+u^4+z)x^4 \\ &\qquad +(3u+2u^3+u^5+uz)x^5+O(x^6). \end{aligned}$$

- The subword $F\overline{F}$: Let

$$\Delta = \Delta(x,z) = 1-(4+z)2x^4+(8z+z^2)x^8.$$

The generating function for partial packing paths in which occurrences of $F\overline{F}$ are marked by z is

$$\begin{aligned} S_{F\overline{F}}(x,u,z) &= \frac{x(3-3x^4z+\sqrt{\Delta})+u\big(1+4x^2+x^4(4-z)-\sqrt{\Delta}\big)}{2x\big(1-2u^2x^2-x^4(2+z)+\sqrt{\Delta}\big)} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(2+u^4)x^4 \\ &\qquad +(3u+2u^3+u^5+uz)x^5+O(x^6). \end{aligned}$$

- The subword $\overline{FU}$: Let

$$\Delta = \Delta(x,z) = 1-2x^4(4+z)+x^8(8+z^2).$$

The generating function for partial packing paths in which occurrences of $\overline{FU}$ are marked by z is

$$\begin{aligned} S_{\overline{FU}}(x,u,z) &= \frac{p(x,u,z)+(u+ux^2-xz-ux^2z)\sqrt{\Delta}}{x(1+z)(-1+2u^2(x^2+x^6(-1+z))+x^4(2+z)-\sqrt{\Delta})} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(2+u^4)x^4+(4u+2u^3+u^5)x^5+O(x^6), \end{aligned}$$

where

$$p(x,u,z) = -u-2x-3ux^2+2x^5+2ux^6-xz-ux^2z-3ux^4z-3ux^6z+x^5z^2+ux^6z^2.$$

- The subword $\overline{F}F$: Let

$$\Delta = \Delta(x,z) = 1-2x^4(4+z)+x^8(8z+z^2).$$

The generating function for partial packing paths in which occurrences of $\overline{F}F$ are marked by z is

$$\begin{aligned} S_{\overline{F}F}(x,u,z) &= \frac{x(3-3x^4z+\sqrt{\Delta})+u(1+4x^2-x^4(-4+z)-4x^6(-1+z)-\sqrt{\Delta})}{2x(1+2u^2x^2(-1+x^4(-1+z))-x^4(2+z)+\sqrt{\Delta})} \\ &= 1+ux+u^2x^2+(2u+u^3)x^3+(2+u^4)x^4+(4u+2u^3+u^5)x^5+O(x^6). \end{aligned}$$

- The subword DU: Let

$$\Delta = \Delta(x,z) = 1 - 2(3+2z)x^4 + x^8(1+2z)^2.$$

The generating function for partial packing paths in which occurrences of DU are marked by z is

$$\begin{aligned} S_{DU}(x,u,z) &= \frac{u(1+x^2)(1+(1+2z)x^2) + x(2+z-z(1+2z)x^4) + (xz-u)\sqrt{\Delta}}{x(1+z)(1-2u^2x^2-(1+2z)x^4+\sqrt{\Delta})} \\ &= 1 + ux + u^2x^2 + (2u+u^3)x^3 + (2+u^4)x^4 \\ &\qquad + (2u+2u^3+u^5+2uz)x^5 + O(x^6). \end{aligned}$$

- The subword $D\overline{D}$: Let

$$\Delta = \Delta(x,z) = 1 - 2(4+z)x^4 + x^8(-4+z)^2.$$

The generating function for partial packing paths in which occurrences of $D\overline{D}$ are marked by z is

$$\begin{aligned} S_{D\overline{D}}(x,u,z) &= \frac{x(3+x^4(-4+z)+\sqrt{\Delta}) + u(1+4x^2+x^4(4-z)-\sqrt{\Delta}}{2x(1-2u^2x^2+x^4(-4+z)+\sqrt{\Delta})} \\ &= 1 + ux + u^2x^2 + (2u+u^3)x^3 + (2+u^4)x^4 \\ &\qquad + (4u+2u^3+u^5)x^5 + O(x^6). \end{aligned}$$

- The subword $\overline{D}D$: Let

$$\Delta = \Delta(x,z) = 1 - 2(3+2z)x^4 + x^8(5+4z).$$

The generating function for partial packing paths in which occurrences of $\overline{D}D$ are marked by z is

$$\begin{aligned} S_{\overline{D}D}(x,u,z) &= \frac{x(1-x^4)(1+2z) + u(1+x^2)(1+x^2(1+2z)) - (u-x)\sqrt{\Delta}}{x(1+z)(1-2u^2x^2-3x^4+\sqrt{\Delta})} \\ &= 1 + ux + u^2x^2 + (2u+u^3)x^3 + (1+u^4+z)x^4 \\ &\qquad + (3u+2u^3+u^5+uz)x^5 + O(x^6). \end{aligned}$$

- The subword $\overline{DF}$: Let

$$\Delta = \Delta(x,z) = 1 - 2(4+z)x^4 + x^8(8+z^2).$$

The generating function for partial packing paths in which occurrences of $\overline{DF}$ are marked by z is

$$\begin{aligned} S_{\overline{DF}}(x,u,z) &= \frac{-x^5(2+z^2) + u(1+3x^4z+2x^2(1+z)-\sqrt{\Delta}) + x(2+z+z\sqrt{\Delta})}{x(1+z)(1-2u^2x^2-x^4(2+z)+\sqrt{\Delta})} \\ &= 1 + ux + u^2x^2 + (2u+u^3)x^3 + (2+u^4)x^4 \\ &\qquad + (3u+2u^3+u^5+uz)x^5 + O(x^6). \end{aligned}$$

3 Bijective results

In this section, we exhibit bijections between sets of packing paths avoiding a subword w (see Table 2) and sets (or subsets) of classical combinatorial objects such as binary words, colored Dyck paths, and Schröder paths. Since the cases considered below are independent, we denote the corresponding bijection by ψ in each case, without affecting clarity. The bijection ψ satisfies $\psi(\epsilon) = \epsilon$.

Let $\mathcal{Q}_n(w)$, $n \geq 0$, be the set of packing paths of width $4n$ avoiding the subword w. We set $\mathcal{Q}(w) = \cup_{n\geq 0}\mathcal{Q}_n(w)$. A *Schröder path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with up steps $u = (1,1)$, horizontal steps $h = (2,0)$, and down steps $d = (1,-1)$ such that the path stays on or above the x-axis. Recall that a Dyck path of semilength n is a Schröder path with no horizontal steps h.

3.1 $\mathcal{Q}_n(\overline{U})$ and binary words

Let P be a nonempty packing path in $\mathcal{Q}_n(\overline{U})$, $n \geq 1$. Then P is of the form

$$P = UF(\overline{F}F)^k DQ$$

where $k \geq 0$ and $Q \in \mathcal{Q}(\overline{U})$. From this decomposition, we recursively construct a binary word $w = w_1 w_2 \cdots w_n$ as follows. If $k = 0$, then $w = 0\psi(Q)$; if $k \neq 0$, then $w = 01^k\psi(Q)$. By deleting the first digit (which is 0) of w, we obtain a bijection between $\mathcal{Q}_n(\overline{U})$ and the set of binary words of length $n-1$. For instance, the image of $UF\overline{F}F\overline{F}FDUFD$ is 110.

3.2 $\mathcal{Q}_n(\overline{U}U)$ and even binary words

Let P be a nonempty packing path in $\mathcal{Q}_n(\overline{U}U)$, $n \geq 1$. Then P is of the form

$$P = UT_0\overline{F}T_1\overline{F}T_2 \cdots \overline{F}T_k DQ$$

where $k \geq 0$, $T_i \in \{F, \overline{UD}\}$ for $0 \leq i \leq k$, and $Q \in \mathcal{Q}(\overline{U}U)$. From this decomposition, we recursively construct a binary word $w = w_1 w_2 \cdots w_{2n}$ as follows. First, we replace each occurrence of F by FF, and each occurrence of $\overline{F}$ by $\overline{FF}$. We then read the word from left to right in blocks of two letters. We initialize $w = \epsilon$. If both letters of the current block are barred or both unbarred, then we add 0 at the end of w; otherwise, we add 1. At the end of the process, w is of length $2n$. Moreover, the form of P guarantees that the number of ones is even, which is equivalent to saying that the number of zeros is even. For instance, the image of $U\overline{UD}DU\overline{UD}DUF\overline{FU}DF\overline{F}FD$ is 1111010010.

3.3 $\mathcal{Q}_n(F)$, $\mathcal{Q}_n(\overline{UD})$, and Schröder paths with no peaks at level 1

From $P \in \mathcal{Q}_n(F)$ we can obtain bijectively a path $Q \in \mathcal{Q}_n(\overline{UD})$ by the local replacement $\overline{UD} \mapsto F$, and this operation is reversible. Hence, we focus on the set $\mathcal{Q}_n(F)$.

Let P be a nonempty packing path in $\mathcal{Q}_n(F)$. Such a path can be uniquely decomposed into the blocks $A = U\overline{U}$, $B = \overline{D}D$, and $C = \overline{DFU}$.

Reading this decomposition from left to right, we construct a Schröder path as follows. We apply the following transformations:

$$A \mapsto u, \qquad B \mapsto d, \qquad C \mapsto h,$$

except when a subword AB starts at height 0 in the block decomposition. In that case, we replace the whole subword $AB = U\overline{UD}D$ by a single horizontal step h.

Note that the blocks A and B have width 2 and are sent to steps of horizontal length 1, whereas the block C has width 4 and is sent to the horizontal step $h = (2, 0)$. Hence a packing path of width $4n$ is sent to a Schröder path of length $2n$. Moreover, the resulting Schröder path has no peak ud at level 1 (the level of a peak is the ordinate of the middle point of the peak). This operation is reversible, and therefore defines a bijection.

3.4 $\mathcal{Q}_n(U\overline{U})$ and Schröder paths with no peaks at odd levels

Let P be a nonempty packing path in $\mathcal{Q}_n(U\overline{U})$, $n \geq 1$. Then P is either of the form

$$P = UF(\overline{F}F)^k DQ \quad \text{where } k \geq 0 \text{ and } Q \in \mathcal{Q}(U\overline{U}), \quad \text{or}$$

$$P = UF(\overline{F}F)^{a_1}\overline{FU}A_1\overline{D}(\overline{F}F)^{a_2}\overline{FU}A_2\overline{D}\cdots(\overline{F}F)^{a_k}\overline{FU}A_k\overline{D}(\overline{F}F)^{a_{k+1}}DQ,$$

where $k \geq 1$, $a_i \geq 0$ for $1 \leq i \leq k+1$, and $A_i \in \mathcal{Q}(U\overline{U})$ (possibly empty) for $1 \leq i \leq k$, and $Q \in \mathcal{Q}(U\overline{U})$.

From this decomposition, we recursively construct a Schröder path as follows. If P satisfies the first case with $k = 0$, i.e., $P = UFDQ$, then we set $\psi(P) = h\psi(Q)$. If P satisfies the first case with $k \geq 1$, then we set $\psi(P) = uh^k d\psi(Q)$. If P satisfies the second case, then we set

$$\psi(P) = uh^{a_1}u\psi(A_1)d\cdots h^{a_{k-1}}u\psi(A_{k-1})dh^{a_k}u\psi(A_k)dh^{a_{k+1}}d\psi(Q).$$

The recursive definition of ψ guarantees that the image of every $P \in \mathcal{Q}_n(U\overline{U})$ is a Schröder path of width $2n$ with no peaks at odd levels. This operation is reversible, and therefore defines a bijection.

3.5 $\mathcal{Q}_n(F\overline{F})$ and colored Dyck paths

In this section, a colored Dyck path is a Dyck path whose up steps come in two colors, denoted by u and $\overline{u}$.

Let P be a nonempty packing path in $\mathcal{Q}_n(F\overline{F})$, $n \geq 1$. Then P has one of the following forms:

$$P = UFDQ \quad \text{where } Q \in \mathcal{Q}(F\overline{F}), \text{ or}$$

$$P = U\overline{U}A_1\overline{DFU}A_2\overline{DF}\cdots\overline{U}A_k\overline{D}DQ \quad \text{where } k \geq 1 \text{ and } A_1, \ldots, A_k, Q \in \mathcal{Q}(F\overline{F}), \text{ or}$$

$$P = U\overline{U}A_1\overline{DFU}A_2\overline{DF}\cdots\overline{U}A_k\overline{DF}FDQ \quad \text{where } k \geq 1 \text{ and } A_1, \ldots, A_k, Q \in \mathcal{Q}(F\overline{F}).$$

From this decomposition, we recursively construct a colored Dyck path as follows. If P satisfies the first case, then we set $\psi(P) = ud\psi(Q)$. If P satisfies the second case, then we set

$$\psi(P) = u^{k-1}\overline{u}\psi(A_1)d\psi(A_2)d\cdots\psi(A_k)d\psi(Q).$$

If P satisfies the third case, then we set

$$\psi(P) = u^{k+1}d\psi(A_1)d\psi(A_2)d\cdots\psi(A_k)d\psi(Q).$$

The recursive definition of ψ guarantees that the image of every $P \in \mathcal{Q}_n(F\overline{F})$ is a colored Dyck path of semilength n. Moreover, $\psi(P)$ starts with ud if P satisfies the first case; it starts with $u^{k-1}\overline{u}$ for the second case; and it starts with $u^{k+1}d$ for the third case. The map ψ is then injective (by induction) and thus defines a bijection from $\mathcal{Q}_n(F\overline{F})$ to the set of colored Dyck paths of semilength n.

3.6 $\mathcal{Q}_n(\overline{F})$, $\mathcal{Q}_n(DU)$, and Schröder paths

Let P be a nonempty packing path in $\mathcal{Q}_n(\overline{F})$, $n \geq 1$. Replace every occurrence of DU with $\overline{F}$. This defines a bijection from $\mathcal{Q}_n(\overline{F})$ to $\mathcal{Q}_n(\overline{DU})$. Hence, we focus on the set $\mathcal{Q}_n(\overline{F})$. A packing path $P \in \mathcal{Q}_n(\overline{F})$ is either of the form

$$P = UFDQ \text{ with } Q \in \mathcal{Q}(\overline{F}), \quad \text{or} \quad P = U\overline{U}Q_1\overline{D}DQ_2 \text{ with } Q_1, Q_2 \in \mathcal{Q}(\overline{F}).$$

From this decomposition, we recursively construct a Schröder path as follows. If P satisfies the first case, then we set $\psi(P) = h\psi(Q)$. If P satisfies the second case, then we set $\psi(P) = u\psi(Q_1)d\psi(Q_2)$.

Notice that $\psi(P)$ is obtained from P by replacing UFD with h, $U\overline{U}$ with u, and $\overline{D}D$ with d. The recursive definition of ψ ensures that the image of every $P \in \mathcal{Q}_n(\overline{F})$ is a Schröder path of length $2n$. The map ψ is clearly injective (by induction) and thus defines a bijection from $\mathcal{Q}_n(\overline{F})$ to the set of Schröder paths of length $2n$.

4 Tables

For the sequence $s_D(n)$ in Table 1, the number $s_D(2n+1)$ corresponds to the n-th term of the Jacobsthal sequence A001045, which also counts compositions of $n+2$ ending with an odd part.

For $n \geq 1$, the sequences in Table 2 satisfy $c_{\overline{F}}(4n) = 2c_F(4n)$, $c_{\overline{UU}}(4n) = c_{\overline{U}}(8n)$, and $c_{\overline{FU}}(4n) = 2c_{UF}(4n)$.

The sum of the first n terms of the sequence corresponding to $U\overline{U}$ and $\overline{D}D$ in Table 4 equals the n-th term of A331793.

Subword w	$s_w(n)$ for $0 \le n \le 15$	OEIS
U	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	A000007
$\overline{U}$	1, 1, 0, 1, 1, 2, 0, 2, 2, 4, 0, 4, 4, 8, 0, 8	New
D	1, 1, 1, 3, 1, 5, 3, 11, 5, 21, 11, 43, 21, 85, 43, 171	A112447
$\overline{D}$	1, 1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, 64, 64, 128	A158780
$F, \overline{U}D$	1, 1, 1, 2, 2, 4, 4, 9, 9, 19, 19, 44, 44, 96, 96, 225	A247623
$\overline{F}, DU$	1, 1, 1, 3, 3, 5, 5, 13, 13, 25, 25, 63, 63, 129, 129, 321	A026003
$U\overline{U}$	1, 1, 0, 1, 1, 2, 1, 4, 3, 7, 4, 15, 11, 27, 16, 59	New
UF	1, 1, 1, 2, 2, 4, 4, 10, 10, 22, 22, 56, 56, 128, 128, 328	New
$\overline{U}U$	1, 1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, 64, 64, 128	A158780
FD	1, 1, 1, 3, 2, 6, 5, 16, 12, 36, 29, 94, 73, 222, 178, 578	New
$F\overline{F}$	1, 1, 1, 3, 3, 6, 6, 17, 17, 39, 39, 110, 110, 267, 267, 753	New
$\overline{FU}$	1, 1, 1, 3, 3, 7, 5, 17, 15, 39, 29, 97, 83, 227, 173, 573	New
$\overline{F}F$	1, 1, 1, 3, 3, 7, 7, 19, 19, 47, 47, 131, 131, 335, 335, 939	New
$D\overline{D}$	1, 1, 1, 3, 3, 7, 7, 19, 19, 47, 47, 123, 123, 311, 311, 803	A026581
$\overline{D}D$	1, 1, 1, 3, 2, 6, 5, 16, 11, 35, 27, 89, 62, 203, 151, 505	New
$\overline{DF}$	1, 1, 1, 3, 3, 6, 6, 17, 17, 37, 37, 102, 102, 233, 233, 633	New

Table 1: Total number of partial packing paths of width n that avoid w.

Subword w	$c_w(4n)$ for $0 \le n \le 10$	OEIS
U, D	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	A000007
$\overline{U}, \overline{D}$	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512	A011782
$F, \overline{U}D$	1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859	A001003
$\overline{F}, DU$	1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718	A006318
$U\overline{U}, \overline{D}D$	1, 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369	A002212
UF, FD	1, 1, 4, 18, 88, 456, 2464, 13736, 78432, 456416, 2697088	A068764
$\overline{U}U, D\overline{D}$	1, 2, 8, 32, 128, 512, 2048, 8192, 32768, 131072, 524288	A081294
$F\overline{F}, \overline{F}F$	1, 2, 8, 40, 224, 1344, 8448, 54912, 366080, 2489344, 17199104	A151374
$\overline{FU}, \overline{DF}$	1, 2, 8, 36, 176, 912, 4928, 27472, 156864, 912832, 5394176	A166229

Table 2: Total number of packing paths of width $4n$ that avoid w.

Subword w	$g_w(n)$ for $2 \leq n \leq 15$	OEIS
U	1, 4, 4, 13, 13, 46, 46, 143, 143, 482, 482, 1475, 1475, 4844	New
$\overline{U}$	1, 2, 3, 7, 12, 31, 40, 100, 133, 366, 439, 1143, 1388, 3919	New
D	0, 0, 2, 2, 4, 10, 22, 38, 58, 154, 240, 504, 678, 1860	New
$\overline{D}$	0, 1, 1, 3, 3, 16, 16, 48, 48, 197, 197, 591, 591, 2192	New
$F, \overline{U}\,\overline{D}$	0, 1, 1, 3, 3, 14, 14, 42, 42, 159, 159, 477, 477, 1676	New
$U\overline{U}$	1, 2, 3, 7, 10, 27, 34, 86, 107, 300, 353, 929, 1088, 3105	New
UF	0, 1, 1, 3, 3, 12, 12, 36, 36, 129, 129, 387, 387, 1320	New
$\overline{U}U$	0, 1, 1, 4, 4, 17, 17, 58, 58, 207, 207, 666, 666, 2243	New
FD	0, 0, 1, 1, 2, 5, 10, 18, 27, 72, 105, 229, 302, 833	New
$F\overline{F}$	0, 0, 0, 1, 1, 4, 4, 15, 15, 54, 54, 175, 175, 598	New
$\overline{F}\,\overline{U}$	0, 0, 0, 0, 2, 4, 6, 14, 26, 66, 86, 214, 300, 814	New
$\overline{F}F$	0, 0, 0, 0, 0, 2, 2, 6, 6, 30, 30, 90, 90, 356	New
$\overline{F}, DU$	0, 0, 0, 2, 2, 8, 8, 32, 32, 116, 116, 390, 390, 1344	New
$D\overline{D}$	0, 0, 0, 0, 0, 2, 2, 6, 6, 38, 38, 114, 114, 516	New
$\overline{D}D$	0, 0, 1, 1, 2, 5, 12, 20, 31, 82, 135, 275, 376, 1027	New
$\overline{D}\,\overline{F}$	0, 0, 0, 1, 1, 4, 4, 17, 17, 62, 62, 215, 215, 746	New

Table 3: Total number of occurrences of w among all partial packing paths of width n.

Subword w	$d_w(4n)$ for $1 \leq n \leq 10$	OEIS
U, D	2, 16, 130, 1072, 8946, 75360, 639570, 5460640, 46853026, 403657456	New
$\overline{U}, \overline{D}$	1, 11, 101, 887, 7689, 66435, 574125, 4968815, 43084817, 374333051	New
$F, \overline{U}\,\overline{D}$	1, 9, 73, 593, 4881, 40665, 342105, 2900385, 24742945, 212155049	A388060
$U\overline{U}, \overline{D}D$	1, 9, 77, 653, 5545, 47265, 404565, 3476485, 29979281, 259330169	New
UF, FD	1, 7, 53, 419, 3401, 28095, 235005, 1984155, 16873745, 144327287	A368773$(2n-1)$
$\overline{U}U, D\overline{D}$	0, 2, 28, 294, 2808, 25770, 232020, 2068430, 18341872, 162178002	New
$F\overline{F}, \overline{F}F$	0, 2, 20, 174, 1480, 12570, 107100, 916230, 7869200, 67827762	2A331793$(n-1)$
$\overline{F}\,\overline{U}, \overline{D}\,\overline{F}$	0, 2, 24, 234, 2144, 19170, 169560, 1492330, 13105536, 115002882	New
$\overline{F}, DU$	0, 4, 44, 408, 3624, 31740, 276660, 2408560, 20974736, 182830644	New

Table 4: Total number of occurrences of w among all packing paths of width $4n$.

5 Use of AI tools

The authors used ChatGPT (OpenAI) to check English grammar and compliance with the journal's formatting guidelines.

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