# Dyck paths with catastrophes modulo the positions of a given pattern

Jean-Luc Baril[1], Sergey Kirgizov[1], and Armen Petrossian[2]

[1]LIB, Université de Bourgogne Franche-Comté
B.P. 47 870, 21078 Dijon Cedex France
E-mail: {barjl,sergey.kirgizov}@u-bourgogne.fr
[2]ESEO Paris
78140 Vélizy-Villacoublay
E-mails: armen.petrossian@eseo.fr

February 23, 2022

**Abstract**

For any pattern $p$ of length at most two, we provide generating functions and asymptotic approximations for the number of $p$-equivalence classes of Dyck paths with catastrophes, where two paths of the same length are $p$-equivalent whenever the positions of the occurrences of the pattern $p$ are the same.

**Keywords:** Dyck path with catastrophes, equivalence relation, pattern, enumeration, generating function.

## 1 Introduction and notation

A *Dyck path with catastrophes* is a lattice path in the first quadrant of the $xy$-plane that starts at the origin, ends on the $x$-axis, and made of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and catastrophe steps $C_k = (1, -k)$, $k \geqslant 2$, so that catastrophe steps always end on the $x$-axis. Depending on the context, we can use the symbol $C$ to design a catastrophe step, and by convenience we use $C_1 = D$. We let $\mathcal{E}$ denote the set of all Dyck paths with catastrophes, and $\mathcal{D}$ be the set of *Dyck paths*, i.e. the paths in $\mathcal{E}$ that do not contain any catastrophe steps $C_k$, $k \geqslant 2$. The *length* $|P|$ of a path $P$ is the number of its steps. The empty path is denoted by $\epsilon$. See Figure 1 for an example of Dyck path with catastrophes of length 14. A *pattern* consists of consecutive steps in a path. We will say that *an occurrence of a pattern* (or for short *a pattern*) is at *position* $i \geqslant 1$ in a path whenever the first step of the pattern appears at the $i$-th step of the path, the second step at the $(i+1)$-th step, and so on. The *height* of an occurrence of a pattern is the minimal ordinate reached by its points. For instance, the Dyck path with catastrophes $P = UUC_2UUUDUDDUC_2UD$ contains three occurrences of the pattern $UU$ at positions 1, 4 and 5, and the heights of these occurrences are respectively 0, 0 and 1.

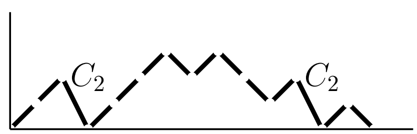



Figure 1: Dyck path with catastrophes $UUC_2UUUDUDDUC_2UD$.

The concept of Dyck path with catastrophes was first introduced by Krinik *et al.* in [9] in the context of queuing theory. They correspond to the evolution of a queue by allowing some resets modeled by a catastrophe step $C_k$, $k \geqslant 2$. Then, Banderier and Wallner [1] provide enumerative results and limit laws of these objects. They show how any non empty path $P \in \mathcal{E}$ can be decomposed either as $P = U\alpha D\beta$, or $P = U\alpha_1 U\alpha_2 \dots U\alpha_k C_k \beta$ for some $k \geqslant 2$, where $\alpha, \alpha_1, \alpha_2, \dots, \alpha_k$ are Dyck paths in $\mathcal{D}$ and $\beta \in \mathcal{E}$. They deduce a functional equation for the generating function $E(x) = \sum_{n \geqslant 0} e_n x^n$ where $e_n$ is the number of paths of length $n$ in $\mathcal{E}$, with the solution

$$E(x) = \frac{2x - 1 + \sqrt{1 - 4\,x^2}}{x - 1 + (1+x)\sqrt{1 - 4\,x^2}}.$$

The sequence $(e_n)_{n \geqslant 0}$ corresponds to A224747 in the On-line Encyclopedia of Integer Sequences (OEIS) [12], and the first values for $n \geqslant 0$ are $1, 0, 1, 3, 5, 12, 23, 52, 105, 232, 480$. More recently, Baril and Kirgizov [4] exhibit a one-to-one correspondence between Dyck paths with catastrophes of length $n$ and Dyck paths of length $2n$ avoiding $UUU$ and $DUD$ at height at least one, and where every occurrence of $UD$ on the $x$-axis appears before (but not necessarily contiguous) an occurrence of $UUU$.

On the other hand, in [2–6, 10] the authors investigate equivalence relations on the sets of Dyck paths, Motzkin paths, skew Dyck paths, Łukasiewicz paths, and Ballot paths where two paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern. The main goal of this study consists in extending these studies for Dyck paths with catastrophes by considering the analogous equivalence relation on $\mathcal{E}$.

**Definition 1** *Two Dyck paths with catastrophes of the same length are $p$-equivalent whenever they have the same positions of the occurrences of the pattern $p$.*

For instance, the path $UUDUUC_3$ is $U$-equivalent to $UUC_2UUC_2$ since the occurrences of $U$ appear at the same positions in the two paths.

In this paper, we provide ordinary generating functions (o.g.f. for short) for the number of $p$-equivalence classes in $\mathcal{E}$ with respect to the length whenever $p$ is a pattern of length at most two (see Table below). Our method consists in providing one-to-one correspondences between equivalence classes and certain subsets of $\mathcal{E}$ (called subsets of *representative elements*) and enumerating them using algebraic techniques. Remark that only one pattern of size 2, namely $DD$, gives a non-rational generating function. For this pattern, the construction of a set of representative elements and its enumeration are quite intricate and handled in the last subsection of the paper.

| Pattern | OEIS [12] | $a_n$, $2 \leqslant n \leqslant 10$ | Theorem | O.g.f. |
|---|---|---|---|---|
| $U$ | Shift of A037952 | 1, 1, 3, 4, 10, 15, 35, 56, 126 | Thm 2 | Algebraic |
| $D$ | New | 1, 1, 3, 5, 11, 21, 42, 84, 162 | Thm 4 | |
| $C$ | A212804 | 1, 1, 2, 3, 5, 8, 13, 21, 34 | Thm 6 | Rational |
| $UU$ | A347493 | 1, 1, 3, 4, 8, 13, 24, 41, 75 | Thm 8 | |
| $UD$ | A215004 | 1, 1, 3, 5, 10, 17, 30, 50, 84 | Thm 10 | |
| $UC$ | New | 1, 1, 2, 4, 5, 9, 15, 24, 40 | Thm 12 | |
| $DC$ | New | 1, 1, 1, 2, 2, 4, 6, 9, 14 | Thm 14 | |
| $CU$ | Shift of A000045 | 1, 1, 1, 2, 3, 5, 8, 13, 21 | Thm 16 | |
| $DU$ | A212804 | 1, 1, 2, 3, 5, 8, 13, 21, 34 | Thm 18 | |
| $DD$ | New | 1, 1, 2, 1, 4, 5, 11, 11, 27 | Thm 21 | Algebraic |

## 2 Patterns of length one

### 2.1 Pattern $U$

Let us define the set $\mathcal{A}$ consisting of the union of the set $\mathcal{D}$ of Dyck paths with the set $\mathcal{D}'$ of paths in $\mathcal{E}$ having only one catastrophe $C_k$, $k \geqslant 2$, located at the end of the path.

**Theorem 1** *There is a bijection between $\mathcal{A}$ and the set of $U$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{A}$ such that $P$ and $Q$ belong to the same $U$-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = P \in \mathcal{A}$. Otherwise, we decompose $P = \alpha_1 C_{k_1} \alpha_2 C_{k_2} \dots \alpha_r C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all $\alpha_i$ do not contain any catastrophe steps. If $\alpha_{r+1}$ is empty, then we set $Q = \alpha_1 D \alpha_2 D \dots \alpha_r C_k$ where $k = (k_1 - 1) + (k_2 - 1) + \dots + (k_{r-1} - 1) + k_r = \sum_{i=1}^{r} k_i - (r-1) \geqslant 2r - (r-1) \geqslant 2$. If $\alpha_{r+1}$ is not empty, it can be written $\alpha' D$, and we set $Q = \alpha_1 D \alpha_2 D \dots \alpha_r D \alpha' C_k$ where $k = k_1 - 1 + k_2 - 1 + k_{r-1} - 1 + k_r - 1 + 1 = \sum_{i=1}^{r} k_i - r + 1 \geqslant 2r - r + 1 \geqslant 2$. In these last two cases, we have $Q \in \mathcal{A}$ so that $P$ and $Q$ belong to the same $U$-equivalence class.

Now, let us prove that if $P$ and $Q$ are two paths of the same length in $\mathcal{A}$ lying in the same $U$-equivalence class, then $P = Q$. According to the decomposition of a Dyck path with catastrophes (see in Introduction), we write either $P = \alpha$ or $P = \alpha U \alpha_1 U \alpha_2 \dots U \alpha_k C_k$ (resp. $Q = \alpha'$ or $Q = \alpha' U \alpha'_1 U \alpha'_2 \dots U \alpha'_\ell C_\ell$) for some $k, \ell \geqslant 2$ and where $\alpha$, $\alpha'$, $\alpha_i$, $\alpha'_i$ are some Dyck paths. Since a Dyck path is characterized by the positions of its up steps, and $P$ and $Q$ are in the same $U$-equivalence class, we necessarily have $\alpha = \alpha'$, $\alpha'_i = \alpha'_i$ for $i \leqslant \min\{k, \ell\}$, which implies that $P = Q$. $\square$

**Theorem 2** *The o.g.f. with respect to the length for the set $\mathcal{A}$ is given by*

$$A(x) = \frac{\left(1 - \sqrt{1 - 4x^2}\right)(1 - x)}{x\left(2x - 1 + \sqrt{1 - 4x^2}\right)}.$$

*The series expansion of $A(x)$ is*

$$1 + x^2 + x^3 + 3x^4 + 4x^5 + 10x^6 + 15x^7 + 35x^8 + 56x^9 + 126x^{10} + 210x^{11} + O\left(x^{12}\right).$$

*Proof.* A path in $\mathcal{A}$ is either a Dyck path in $\mathcal{D}$, or a path $\alpha U\alpha_1 U\alpha_2 U\alpha_3 \dots U\alpha_k C_k$ where $k \geqslant 2$ and $\alpha, \alpha_1, \dots, \alpha_k$ are some Dyck paths. We deduce that $A(x) = D(x) + D(x) \cdot \frac{x^2 D(x)^2}{1-xD(x)} \cdot x$ where $D(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$ is the generating function for Dyck paths (see [7]). □

**Remark 1** *We have* $x^2 - 2x + 1 + (-2x^2 + 3x - 1)A(x) + (2x^2 - x)A(x)^2 = 0$, *and the coefficient* $a_n$ *of* $x^n$ *in* $A(x)$ *is* $\binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$ *which generates a shift of the sequence A037952 in OEIS [12]. Using a classical analysis ( [8, 11]) of dominant singularity of* $A(x)$, $a_n$ *has the asymptotic approximation* $\frac{2^n}{\sqrt{2\pi n}}$.

## 2.2 Pattern $D$

Let us define the set $\mathcal{B}$ consisting of the union of the set $\mathcal{D}$ of Dyck paths with the set $\mathcal{D}''$ of paths $P \in \mathcal{E}$ having only one catastrophe.

**Theorem 3** *There is a bijection between* $\mathcal{B}$ *and the set of* $D$*-equivalence classes of* $\mathcal{E}$.

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{B}$ such that $P$ and $Q$ belong to the same $D$-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = P \in \mathcal{B}$; otherwise, we decompose $P = \alpha_1 C_{k_1} \alpha_2 C_{k_2} \dots \alpha_r C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all $\alpha_i$ do not contain any catastrophe steps. If $\alpha_{r+1}$ is empty, then we set $Q = \alpha_1 U \alpha_2 U \dots \alpha_r C_k$ where $k = (k_1 + 1) + (k_2 + 1) + \dots + (k_{r-1} + 1) + k_r = r - 1 + \sum_{i=1}^r k_i$. If $\alpha_{r+1}$ is not empty, it is necessarily a Dyck path, and we set $Q = \alpha_1 U \alpha_2 U \dots \alpha_r C_k \alpha_{r+1}$ where $k = k_1 + 1 + k_2 + 1 + k_{r-1} + 1 + k_r = r - 1 + \sum_{i=1}^r k_i$. In all these cases, we have $Q \in \mathcal{B}$ so that $P$ and $Q$ belong to the same $D$-equivalence class.

With a similar argument as for the proof of Theorem 2, it is easy to prove that if $P$ and $Q$ are two paths in $\mathcal{B}$ lying in the same class, then $P = Q$. □

**Theorem 4** *The o.g.f. with respect to the length for the set* $\mathcal{B}$ *is given by*

$$B(x) = \frac{\left(1 - \sqrt{1-4x^2}\right)\left(2x^3 - 4x^2 + 1 + (2x^2 - 1)\sqrt{1-4x^2}\right)}{2x^4\left(2x - 1 + \sqrt{1-4x^2}\right)}.$$

*The series expansion of* $B(x)$ *is*

$$1 + x^2 + x^3 + 3x^4 + 5x^5 + 11x^6 + 21x^7 + 42x^8 + 84x^9 + 162x^{10} + 330x^{11} + O\left(x^{12}\right).$$

*Proof.* A path in $\mathcal{B}$ is either a Dyck path in $\mathcal{D}$, or a path of the form $\alpha U\alpha_1 U\alpha_2 U\alpha_3 \dots U\alpha_k C_k\beta$ where $k \geqslant 2$ and $\alpha, \alpha_1, \dots, \alpha_k, \beta$ are some Dyck paths. We deduce that $B(x)$ satisfies the functional equation $B(x) = D(x) + \frac{x^3 D(x)^4}{1-xD(x)}$ where $D(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$ is the o.g.f. for Dyck paths. □

**Remark 2** *We have* $5x^3 - 4x^2 - x + 1 + (-2x^4 - 5x^3 + 5x^2 + x - 1)B(x) + (2x^5 - x^4)B(x)^2 = 0$, *and the coefficient* $b_n$ *of* $x^n$ *in* $B(x)$ *is*

$$b_n = \begin{cases} \binom{n}{\frac{n-3}{2}} & \text{if } n \text{ is odd,} \\ \binom{n}{\frac{n-4}{2}} + \binom{n}{\frac{n}{2}} / \left(\frac{n}{2} + 1\right) & \text{otherwise.} \end{cases}$$

*This last result can be easily obtained by combining already known formulas (see A002054 and A344191 in [12]), or directly using Maple. The sequence* $b_n$ *has an asymptotic approximation* $\frac{2^{n+\frac{1}{2}}}{\sqrt{\pi n}}$.

## 2.3 Pattern $C$

In this section, two paths are $C$-equivalent whenever they coincide on all their catastrophe steps $C_k$ for $k \geqslant 2$. For instance, $UUDDUUC_2UUUC_3UUC_2UD$ is $C$-equivalent to $UDUDUUC_2UUUC_3UUC_2UD$, while it is not $C$-equivalent with $UUUDUUC_4UUUC_3UUC_2UD$.

Let $\mathcal{C}$ be the set of paths $P \in \mathcal{E}$ such that:
$(i)$ $P = (UD)^k$ for $k \geqslant 0$, or
$(ii)$ $P = (UD)^{\ell_1}U^{k_1}C_{k_1}(UD)^{\ell_2}U^{k_2}C_{k_2} \dots (UD)^{\ell_r}U^{k_r}C_{k_r}(UD)^{\ell_{r+1}}$ with $r \geqslant 1$, $\ell_i \geqslant 0$ for $1 \leqslant i \leqslant r+1$, and $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$.

**Theorem 5** *There is a bijection between* $\mathcal{C}$ *and the set of* $C$*-equivalence classes of* $\mathcal{E}$.

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{C}$ such that $P$ and $Q$ belong to the same $C$-equivalence class. If we have $P \in \mathcal{D}$, then we choose $Q = (UD)^k \in \mathcal{C}$ with $k = |P|/2$. Otherwise, we decompose $P = \alpha_1 C_{k_1}\alpha_2 C_{k_2} \dots \alpha_r C_{k_r}\alpha_{r+1}$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$ and all $\alpha_i$ do not contain any catastrophe steps. We set $Q = (UD)^{j_1}U^{k_1}C_{k_1}(UD)^{j_2}U^{k_2}C_{k_2} \dots (UD)^{j_r}U^{k_r}C_{k_r}(UD)^{j_{r+1}}$ where $j_i + k_i = |\alpha_i|$, $1 \leqslant i \leqslant r$ and $j_{r+1} = |\alpha_{r+1}|$. We have $Q \in \mathcal{C}$ so that $P$ and $Q$ belong to the same $C$-equivalence class. Due to the form of $Q$, it is straightforward to see that if $P$ and $Q$ are two paths in $\mathcal{C}$ lying in the same class, then $P = Q$. □

**Theorem 6** *The o.g.f. with respect to the length for the set $\mathcal{C}$ is given by*

$$C(x) = \frac{1-x}{1-x-x^2}.$$

*The series expansion of $C(x)$ is*

$$1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + 55x^{11} + O\left(x^{12}\right).$$

*Proof.* Due to the definition of the set $\mathcal{C}$, we obtain directly the functional equation $C(x) = \frac{1}{1-x^2} + \frac{1}{1-x^2} \cdot \frac{C'(x)}{1-C'(x)}$ where $C'(x) = \frac{x^3}{1-x} \cdot \frac{1}{1-x^2}$ is the o.g.f. for the paths of the form $(UD)^\ell U^k C_k$ for $\ell \geqslant 0$ and $k \geqslant 2$. □

**Remark 3** *The coefficient $c_n$ of $x^n$ in $C(x)$ satisfies $c_n = c_{n-1} + c_{n-2}$ for $n \geqslant 2$ with $c_0 = 1$ and $c_1 = 0$, which generates the sequence A212804 in [12] (it is a variant of the well known Fibonacci sequence A000045). Using the classical method for asymptotic approximation (see [8, 11]), we have*

$$c_n \sim \frac{3-\sqrt{5}}{5-\sqrt{5}} \cdot \left(\frac{\sqrt{5}-1}{2}\right)^n.$$

# 3 Patterns of length two

## 3.1 Pattern $UU$

Let $\mathcal{F}$ be the set of paths $P \in \mathcal{E}$ such that:
*(i)* $P = (UD)^k$ for $k \geqslant 0$, or
*(ii)* $P = (UD)^{\ell_0} U^{k_1} \alpha_1 U^{k_2} \alpha_2 \ldots U^{k_r} \alpha_r$ where $r \geqslant 1$, $\ell_0 \geqslant 0$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$, all $\alpha_i$ for $1 \leqslant i \leqslant r-1$ are either $(DU)^k D$ or $(DU)^k DD$ for some $k \geqslant 0$, and $\alpha_r$ is either $(DU)^k C_s$ or $(DU)^k DC_s$ for some $k \geqslant 0$ and with $s \geqslant 1$ is so that the path ends on the $x$-axis (note that $s$ can be 1, and in this case $C_1 = D$).

**Theorem 7** *There is a bijection between $\mathcal{F}$ and the set of $UU$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{F}$ such that $P$ and $Q$ belong to the same $UU$-equivalence class. If $P$ does not contain occurrences of $UU$, then we choose $Q = (UD)^k \in \mathcal{F}$ where $k = |P|/2$. Otherwise, we decompose $P = \alpha_0 U^{k_1} \alpha_1 U^{k_2} \dots \alpha_{r-1} U^{k_r} \alpha_r$ where $r \geqslant 1$, $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$, and such that all occurrences of $UU$ in $P$ belong necessarily to a run $U^{k_i}$ for some $i$.

We set $Q = (UD)^{\ell_0} U^{k_1} \beta_1 U^{k_2} \dots \beta_{r-1} U^{k_r} \beta_r$ where $\ell_0 = |\alpha_0|$, and for $1 \leqslant i \leqslant r-1$, $\beta_i = (DU)^{\frac{t_i-1}{2}} D$ if $t_i = |\alpha_i|$ is odd, $\beta_i = (DU)^{\frac{t_i-2}{2}} DD$ otherwise; finally, we set $\beta_r = (DU)^{\frac{t_r-1}{2}} C_s$ if $t_r = |\alpha_r|$ is odd, and otherwise, $\beta_r = (DU)^{\frac{t_r-2}{2}} DC_s$ with $s \geqslant 1$ is chosen so that $Q$ ends on the $x$-axis.

We have $Q \in \mathcal{F}$ so that $P$ and $Q$ belong to the same $UU$-equivalence class. It is straightforward to see that if $P$ and $Q$ are two paths in $\mathcal{F}$ lying in the same class, then $P = Q$. □

**Theorem 8** *The o.g.f. with respect to the length for the set $\mathcal{F}$ is given by*

$$F(x) = \frac{x-1}{(x+1)(x^3 - x^2 + 2x - 1)}.$$

*The series expansion of $F(x)$ is*

$$1 + x^2 + x^3 + 3x^4 + 4x^5 + 8x^6 + 13x^7 + 24x^8 + 41x^9 + 73x^{10} + 127x^{11} + O\left(x^{12}\right).$$

*Proof.* Due to the definition of $\mathcal{F}$, we have $F(x) = \frac{1}{1-x^2} + \frac{1}{1-x^2} \cdot \frac{1}{1-\frac{x^3}{(1-x)^2}} \frac{x^3}{(1-x)^2}$. A simple calculation completes the proof. □

**Remark 4** *The coefficient $f_n$ of $x^n$ in $F(x)$ satisfies $f_n = f_{n-1} + f_{n-2} + f_{n-4}$ with $f_0 = 1, f_1 = 0, f_2 = 1, f_3 = 1$, which generates the sequence A347493 in [12]. An asymptotic approximation of $f_n$ is*

$$f_n \sim \frac{(a-1)(a^2-a+2)^n}{a(a+1)(-3a^2+2a-2)} \approx 0.2621257657 \cdot \mathrm{e}^{0.5623991485 \cdot n},$$

*where* $a = \frac{(44+12\sqrt{69})^{2/3} + 2(44+12\sqrt{69})^{1/3} - 20}{6(44+12\sqrt{69})^{1/3}}$.

## 3.2 Pattern $UD$

Let $\mathcal{G}$ be the set of paths $P \in \mathcal{E}$ such that either :
*(i)* $P = (UD)^k$ for some $k \geqslant 0$, or
*(ii)* $P = (UD)^\ell U^k C_k (UD)^m$ with $\ell, m \geqslant 0$ and $k \geqslant 2$, or
*(iii)* $P = (UD)^{\ell_0} U^{k_1} (UD)^{\ell_1} U^{k_2} (UD)^{\ell_2} \dots U^{k_r} (UD)^{\ell_r} U^{k_{r+1}} C_s (UD)^{\ell_{r+1}}$ with $r \geqslant 1$, $\ell_0, \ell_{r+1} \geqslant 0$, $k_i \geqslant 1$ for $1 \leqslant i \leqslant r$, $k_{r+1} \geqslant 0$, $\ell_i \geqslant 1$ for $1 \leqslant i \leqslant r$, and $s \geqslant 1$ is so that the path ends on the $x$-axis (note that $s$ can be one).

**Theorem 9** *There is a bijection between $\mathcal{G}$ and the set of $UD$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{G}$ such that $P$ and $Q$ belong to the same $UD$-equivalence class. If $P$ satisfies the case $(i)$, then we obviously set $Q = P$. If $P = (UD)^{\ell}\alpha(UD)^m$ with $l, m \geqslant 0$ and $\alpha$ is a nonempty path in $\mathcal{E}$ avoiding $UD$, then we set $Q = (UD)^{\ell}U^kC_k(UD)^m$. Otherwise, we decompose $P = (UD)^{\ell_0}\alpha_1(UD)^{\ell_1}\alpha_2(UD)^{\ell_2}\ldots\alpha_r(UD)^{\ell_r}\alpha_{r+1}$ with all $\alpha_i$ are nonempty partial paths avoiding $UD$ (except $\alpha_{r+1}$ that can be empty). If $|\alpha_{r+1}| > 1$ then we set $Q = (UD)^{\ell_0}U^{k_1}(UD)^{\ell_1}U^{k_2}(UD)^{\ell_2}\ldots$ $U^{k_r}(UD)^{\ell_r}U^{k_{r+1}}C_s$ where $k_i = |\alpha_i|$ and $k_{r+1} + 1 = |\alpha_{r+1}|$; otherwise, if $|\alpha_{r+1}| = 0$ then we set $Q = (UD)^{\ell_0}U^{k_1}(UD)^{\ell_1}U^{k_2}(UD)^{\ell_2}\ldots U^{k_r-1}C_s(UD)^{\ell_r}$ where $k_i = |\alpha_i|$ and $s$ is so that $Q$ ends on the $x$-axis. For all these cases, we have $Q \in \mathcal{G}$ so that $P$ and $Q$ belong to the same $UD$-equivalence class. Due to the form of a path in $\mathcal{G}$, if $P$ and $Q$ are two paths in $\mathcal{G}$ lying in the same $UD$-equivalence class, the $P = Q$. □

**Theorem 10** *The o.g.f. with respect to the length for the set $\mathcal{G}$ is given by*

$$G(x) = \frac{2x^3 - 2x + 1}{(1 - x^2 - x)\,(1 - x)^2\,(x + 1)}.$$

*The series expansion of $G(x)$ is*

$$1 + x^2 + x^3 + 3x^4 + 5x^5 + 10x^6 + 17x^7 + 30x^8 + 50x^9 + 84x^{10} + 138x^{11} + O\left(x^{12}\right).$$

*Proof.* According to the different cases in the definition of $\mathcal{G}$, $G(x)$ is given by:

$$1 + \frac{x^2}{1 - x^2} + \frac{1}{1 - x^2}\frac{x^3}{1 - x}\frac{1}{1 - x^2} + \frac{1}{1 - x^2}\frac{R(x)}{1 - R(x)}\frac{x}{1 - x}\frac{1}{1 - x^2}$$

where $R(x) = \frac{x^3}{(1-x)(1-x^2)}$ is the o.g.f. of paths $U^k(UD)^{\ell}$ with $k, \ell \geqslant 1$. □

**Remark 5** *The coefficient $g_n$ of $x^n$ in $G(x)$ satisfies $g_n = 2g_{n-1} + g_{n-2} - 3g_{n-3} + g_{n-5}$ for $n > 4$ with $g_1 = 0$, $g_0 = g_2 = g_3 = 1$, $g_4 = 3$, which generates the sequence A215004 in [12]. An asymptotic approximation of $g_n$ is*

$$g_n \sim \frac{4(5 - 2\sqrt{5})}{5(\sqrt{5} - 3)^2} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^n \approx 0.7236067987 \cdot \mathrm{e}^{0.4812118252 \cdot n}.$$

### 3.3 Pattern $UC$

In this section, two paths are $UC$-equivalent whenever they coincide on all their occurrences $UC_k$ for $k \geqslant 2$. For instance, $UUDUC_2$ and $UDUUC_2$ are $UC$-equivalent, while $UDUUC_2$ and $UUUUC_4$ are not.

Let $\mathcal{I}_1$ be the set of paths of length $n \geqslant 0$, $n \notin \{1,3\}$, defined by either
$(i)$ $(UD)^{\frac{n}{2}}$ if $n$ is even, or
$(ii)$ $(UD)^{\frac{n-5}{2}}UUUDC_2$ if is $n$ is odd.
Let $\mathcal{I}$ be the set consisting of the union of $\mathcal{I}_1$ with the set of paths of the form $\alpha_1 U^{k_1} C_{k_1} \alpha_2 U^{k_2} C_{k_2} \dots \alpha_r U^{k_r} C_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, all values $k_i$ are at least 2, and all $\alpha_i$ are in $\mathcal{I}_1$.

**Theorem 11** *There is a bijection between $\mathcal{I}$ and the set of $UC$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{I}$ such that $P$ and $Q$ belong to the same $UC$-equivalence class. If $P$ does not contain any pattern $UC$, then we set $Q = (UD)^{\frac{n}{2}}$ if $n$ is even, and $Q = (UD)^{\frac{n-5}{2}}UUUDC_2$ otherwise. If $P$ contains $r \geqslant 1$ occurrences of $UC$, then we decompose $P = \alpha_1 UC \alpha_2 UC \dots \alpha_r UC \alpha_{r+1}$ where all $\alpha_i$ are paths avoiding $UC$. Showing the size of every catastrophes, we obtain the decomposition $P = \alpha_1 UC_{k_1} \alpha_2 UC_{k_2} \dots \alpha_r UC_{k_r} \alpha_{r+1}$ where $k_i \geqslant 2$ for $1 \leqslant i \leqslant r$.

We set $Q = \beta_1 U^{k_1} C_{k_1} \beta_2 U^{k_2} C_{k_2} \dots \beta_r U^{k_r} C_{k_r} \beta_{r+1}$ where all $\beta_i$ are in $\mathcal{I}_1$ (all $\beta_i$ are entirely determined by the length of each $\alpha_i$). We have $Q \in \mathcal{I}$ so that $P$ and $Q$ belong to the same $UC$-equivalence class. Obviously, due to the definition of $\mathcal{I}$, if $P$ and $Q$ belong to $\mathcal{I}$ in the same $UC$-equivalence class, then $P = Q$. □

**Theorem 12** *The o.g.f. with respect to the length for the set $\mathcal{I}$ is given by*

$$I(x) = \frac{x^5 - 2\,x^4 + 2\,x^3 - 2\,x^2 + 2\,x - 1}{x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + 2\,x - 1}.$$

*The series expansion of $I(x)$ is*

$$1 + x^2 + x^3 + 2x^4 + 4x^5 + 5x^6 + 9x^7 + 15x^8 + 24x^9 + 40x^{10} + 65x^{11} + O\left(x^{12}\right).$$

*Proof.* The o.g.f. for $\mathcal{I}_1$ is $I_1(x) = \frac{1}{1-x} - x - x^3$ and the o.g.f. for $\mathcal{I}$ is $I_1(x) + I_1(x) \cdot \frac{I_2(x)}{1-I_2(x)}$ where $I_2(x) = \frac{I_1(x)x^3}{1-x}$ is the o.g.f. of a nonempty sequence of terms of the form $\alpha U^k C_k$ with $\alpha \in \mathcal{I}_1$ and $k \geqslant 2$. □

**Remark 6** *The coefficient $i_n$ of $x^n$ in $I(x)$ satisfies $i_n = 2i_{n-1} - i_{n-2} + i_{n-3} - i_{n-4} + i_{n-5} - i_{n-6} + i_{n-7}$ for $n \geqslant 7$ with $i_0 = 1, i_1 = 0, i_2 = 1, i_3 = 1, i_4 = 2, i_5 = 4$, and $i_6 = 5$. This sequence does not appear in [12]. An asymptotic approximation of $i_n$ is*

$$i_n \sim 0.2813451087 \cdot \mathrm{e}^{0.4951400086 \cdot n}.$$

### 3.4 Pattern $DC$

In this section, two paths are $DC$-equivalent whenever they coincide on all their occurrences $DC_k$ for $k \geqslant 2$.

Let $\mathcal{J}_1$ be the set of paths of length $n \geqslant 0$, $n \neq 1$, defined by either
(*i*) $(UD)^{\frac{n}{2}}$ if $n$ is even, or
(*ii*) $(UD)^{\frac{n-3}{2}} UUC_2$ if is $n$ is odd.
Let $\mathcal{J}$ be the set consisting of the union of $\mathcal{J}_1$ with the set of paths of the form $\alpha_1 U^{k_1+1} DC_{k_1} \alpha_2 U^{k_2+1} DC_{k_2} \ldots \alpha_r U^{k_r+1} DC_{k_r} \alpha_{r+1}$ where $r \geqslant 1$, all values $k_i$ are at least 2, and all $\alpha_i$ are in $\mathcal{J}_1$.

**Theorem 13** *There is a bijection between $\mathcal{J}$ and the set of $DC$-equivalence classes of $\mathcal{E}$.*

*Proof.* The proof is obtained *mutatis mutandis* as for Theorem 11. □

**Theorem 14** *The o.g.f. with respect to the length for the set $\mathcal{J}$ is given by*

$$J(x) = \frac{x^3 - 2x^2 + 2x - 1}{x^7 - x^6 + x^5 - x^2 + 2x - 1}.$$

*The series expansion of $J(x)$ is*

$$1 + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 4x^7 + 6x^8 + 9x^9 + 14x^{10} + 20x^{11} + O\left(x^{12}\right).$$

*Proof.* The o.g.f. for $\mathcal{J}_1$ is $J_1(x) = \frac{1}{1-x} - x$ and the o.g.f. for $\mathcal{J}$ is $J_1(x) + J_1(x) \cdot \frac{J_2(x)}{1-J_2(x)}$ where $J_2(x) = \frac{J_1(x)x^5}{1-x}$ is the o.g.f. of a nonempty sequence of terms of the form $\alpha U^{k+1} DC_k$ with $\alpha \in \mathcal{J}_1$ and $k \geqslant 2$. □

**Remark 7** *The coefficient $j_n$ of $x^n$ in $J(x)$ satisfies $j_n = 2j_{n-1} - j_{n-2} + j_{n-5} - j_{n-6} + j_{n-7}$ for $n \geqslant 7$ with $j_0 = 1, j_1 = 0, j_2 = j_3 = j_4 = 1, j_5 = 2, j_6 = 2$ which is not in [12]. An asymptotic approximation of $j_n$ is*

$$j_n \sim 0.2531721798 \cdot \mathrm{e}^{0.3967526042 \cdot n}.$$

### 3.5 Pattern $CU$

In this section, two paths are $CU$-equivalent whenever they coincide on all their occurrences $C_kU$ for $k \geqslant 2$.

Let $\mathcal{K}_1$ be the set of paths of even length defined by $(UD)^k$ for $k \geqslant 0$. Let $\mathcal{K}_2$ be the set of paths of length $n \neq 1$ defined by $(UD)^k$, $k \geqslant 1$, if $n$ is even, and $(UD)^k UUC_2$, $k \geqslant 0$, if $n$ is odd.
Let $\mathcal{K}$ be the set consisting of the union of $\mathcal{K}_2$ with the set of paths of the form $\beta_1 U^{k_1} C_{k_1} \beta_2 U^{k_2} C_{k_2} \dots \beta_r U^{k_r} C_{k_r} \beta_{r+1}$ where $k_i \geqslant 2$ for $1 \leqslant i < r$, and for $1 \leqslant i < r+1$, $\beta_i \in \mathcal{K}_1$, and $\beta_{r+1}$ is a path from $\mathcal{K}_2$.

**Theorem 15** *There is a bijection between $\mathcal{K}$ and the set of $CU$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{K}$ such that $P$ and $Q$ belong to the same $CU$-equivalence class. If $P$ does not contain any pattern $CU$, then we set $Q = (UD)^{\frac{n}{2}}$ whenever $n = |P|$ is even, and we set $Q = (UD)^{\frac{n-3}{2}} UUC_2$ otherwise; in both cases $Q \in \mathcal{K}_2$. If $P$ contains $r \geqslant 1$ occurrences of $CU$, then we decompose $P = \alpha_1 CU \alpha_2 CU \dots \alpha_r CU \alpha_{r+1}$ where all $\alpha_i$ are paths avoiding $CU$. Showing the size of every catastrophe, we obtain the decomposition $P = \alpha_1 C_{k_1} U \alpha_2 C_{k_2} U \dots \alpha_r C_{k_r} U \alpha_{r+1}$ where all $k_i$ are at least 2. So, we set $Q = \beta_1 U^{k_1} C_{k_1} \beta_2 U^{k_2} C_{k_2} \dots \beta_r U^{k_r} C_{k_r} \beta_{r+1}$ where $|\alpha_1| = |\beta_1|$, $|\alpha_i| + 1 = |\beta_i| + k_i$ for $1 < i < r+1$, $|\alpha_{r+1}| + 1 = |\beta_{r+1}|$, and

- for $1 \leqslant i < r+1$, $\beta_i \in \mathcal{K}_1$;
- $\beta_{r+1}$ is a path from $\mathcal{K}_2$.

We have $Q \in \mathcal{K}$ so that $P$ and $Q$ belong to the same $CU$-equivalence class. Obviously, due to the definition of $\mathcal{K}$, if $P$ and $Q$ belong to $\mathcal{K}$ in the same $CU$-equivalence class, then $P = Q$. □

**Theorem 16** *The o.g.f. with respect to the length for the set $\mathcal{K}$ is given by*

$$K(x) = \frac{x^4 + x - 1}{x^2 + x - 1}.$$

*The series expansion of $K(x)$ is*

$$1 + x^2 + x^3 + x^4 + 2x^5 + 3x^6 + 5x^7 + 8x^8 + 13x^9 + 21x^{10} + 34x^{11} + O\left(x^{12}\right).$$

*Proof.* Due to the definition of $\mathcal{K}$, the o.g.f. $K(x)$ is given by

$$K(x) = \frac{1}{1-x} - x + \frac{1}{1-x^2} \cdot \frac{1}{1 - \frac{x^3}{(1-x)(1-x^2)}} \cdot \frac{x^3}{1-x} \cdot \frac{x^2}{1-x}.$$

□

**Remark 8** *The coefficient $k_n$ of $x^n$ in $K(x)$ satisfies $k_n = k_{n-1} + k_{n-2}$ for $n \geqslant 5$ with $k_0 = 1, k_1 = 0, k_2 = k_3 = k_4 = 1$ which is a shift of the Fibonacci sequence A000045 in [12]. An asymptotic approximation of $k_n$ is*

$$k_n \sim \frac{2(\sqrt{5}-2)}{5-\sqrt{5}} \cdot \left(\frac{2}{\sqrt{5}-1}\right)^n \approx 0.1708203931 \cdot \mathrm{e}^{0.4812118251}.$$

## 3.6 Pattern $DU$

Let $\mathcal{L}$ be set of paths $P \in \mathcal{E}$ such that $P = U^{\ell_1}(DU)^{k_1}U^{\ell_2}(DU)^{k_2} \dots U^{k_r}(DU)^{k_r}U^{\ell_{r+1}}C_s$ with $r \geqslant 1$, $\ell_i, k_i \geqslant 1$ for $1 \leqslant i \leqslant r$, $\ell_{r+1} \geqslant 0$, and $s$ is so that the path ends on the $x$-axis (note that $s$ can be 1, and in this case $C_1 = D$).

**Theorem 17** *There is a bijection between $\mathcal{L}$ and the set of $DU$-equivalence classes of $\mathcal{E}$.*

*Proof.* First, we prove that for every $P \in \mathcal{E}$ there is $Q \in \mathcal{G}$ such that $P$ and $Q$ belong to the same $DU$-equivalence class. We decompose $P = \alpha_1(DU)^{k_1}\alpha_2(DU)^{k_2}\alpha_3(DU)^{k_3} \dots \alpha_r(DU)^{k_r}\alpha_{r+1}$ with all $\alpha_i$ are nonempty paths avoiding $DU$, with $r \geqslant 0$, and all $k_i$ are at least one. Then we set $Q = U^{\ell_1}(DU)^{k_1}U^{\ell_2}(DU)^{k_2}U^{\ell_3}(DU)^{k_3} \dots U^{\ell_r}(DU)^{k_r}U^{\ell_{r+1}-1}C_s$ where $\ell_i = |\alpha_i|$, and where $s$ is so that the path ends on the $x$-axis. For all these cases, we have $Q \in \mathcal{L}$ so that $P$ and $Q$ belong to the same $DU$-equivalence class. Due to the form of a path in $\mathcal{L}$, if $P$ and $Q$ are two paths in $\mathcal{L}$ lying in the same $DU$-equivalence class, the $P = Q$. □

**Theorem 18** *The o.g.f. with respect to the length for the set $\mathcal{L}$ is given by*

$$L(x) = \frac{1-x}{1-x-x^2}.$$

*The series expansion of $L(x)$ is*

$$1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + 55x^{11} + O\left(x^{12}\right).$$

*Proof.* According to the definition of $\mathcal{L}$ and handling separately the cases where $r = 1$ and $r \geqslant 2$, the o.g.f. $L(x)$ is given by:

$$1 + \frac{x^2}{1-x^2} + \frac{R(x)}{1-R(x)} \cdot \frac{x}{1-x}$$

where $R(x) = \frac{x^3}{(1-x)(1-x^2)}$ is the o.g.f. of paths $U^\ell(DU)^k$ with $k, \ell \geqslant 1$. □

**Remark 9** *The coefficient $\ell_n$ of $x^n$ in $G(x)$ satisfies $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geqslant 3$ with $\ell_0 = 1, \ell_1 = 0, \ell_2 = 1$, which generates the sequence A212804 in [12] (a variant of the well known Fibonacci sequence A000045). An asymptotic approximation of $\ell_n$ is*

$$\ell_n \sim \frac{(\sqrt{5}-3)}{5-\sqrt{5}} \cdot \left(\frac{2}{\sqrt{5}-1}\right)^n \approx 0.2763932027 \cdot \mathrm{e}^{0.4812118251}.$$

### 3.7 Pattern $DD$

In [2], the authors exhibit a set $\overline{\mathcal{A}}$ (denoted $\mathcal{A}$ in [2]) of representatives for the $DD$-equivalence classes of Dyck paths: $\overline{\mathcal{A}}$ consists of Dyck paths $P$ satisfying

**Condition** $(C)$ : *$P$ avoids $UUDU$ and the height of every occurrence of $UDU$ is at most one.*

For a Dyck path $P$, we denote by $\psi(P)$ the unique path in $\overline{\mathcal{A}}$ $DD$-equivalent to $P$. In [2], it is shown how $\psi(P)$ can be constructed from $P$ after several transformations preserving the positions of all occurrences of $DD$. The following proposition extends this result for the set of Dyck paths with only one catastrophe $C_k$ at the end.

**Proposition 1** *For any Dyck path $P \in \mathcal{E}$ with only one catastrophe $C_k$, $k \geqslant 2$, at the end, there is a unique Dyck path with only one catastrophe $C_k$ at the end, denoted $\psi(P)$, satisfying Condition $(C)$ and equivalent to $P$.*

*Proof.* Let $Q$ be the Dyck path obtained from $P$ by replacing $C_k$ with $D^k$. Using [2], there is a unique path $\psi(Q)$ in $\overline{\mathcal{A}}$ in the class of $Q$. Replacing the last $D^k$ of $\psi(Q)$ with $C_k$, there is a unique Dyck path with only catastrophe $C_k$ at the end equivalent to $P$. □

**Lemma 1** *Let $P$ be a path in $\mathcal{E}$. If $P$ contains at least two catastrophes, then there exists $Q \in \mathcal{E}$ in the class of $P$, such that $Q$ has only one catastrophe.*

*Proof.* We obtain $Q$ from $P$ by replacing with $U$ every catastrophe, except the last, and by increasing the size of the last catastrophe so that $Q \in \mathcal{E}$. See Figure 2 for an example with two catastrophes. □

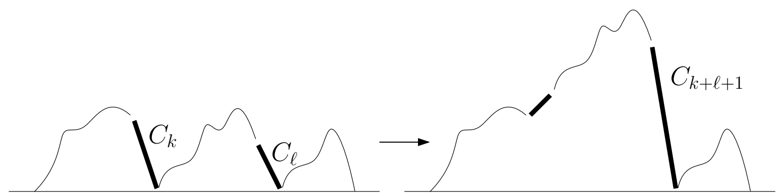



Figure 2: Example for Lemma 1 with two catastrophes.

Considering this previous lemma, we can focus our study on the set of Dyck paths with at most one catastrophe.

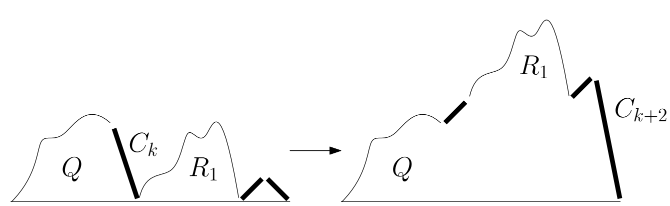
**Lemma 2** *Let $P$ be a $n$-length path in $\mathcal{E}$. If $P$ contains exactly one catastrophe, then $n$ and the size of the catastrophe have not the same parity.*

*Proof.* For any Dyck path with exactly one catastrophe of size $k$, the number of $U$ minus the number of $D$ equals $k$, which implies that the number of $U$ and $D$ (which is equal to $n-1$) has the same parity as $k$. □

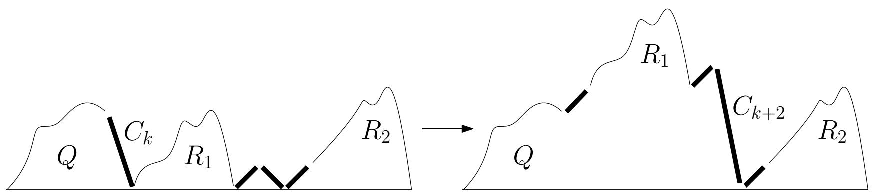
**Lemma 3** *Let $P$ be a path in $\mathcal{E}$ with only one catastrophe $C_k$, $k \geqslant 2$. We decompose $P = QC_kR$ where $R$ is a Dyck path. Let us assume that $R$ contains isolated $D$-steps (i.e. step $D$ that does not lie in an occurrence $DD$). Then, there is a path in $\mathcal{E}$ with only one catastrophe $C_{k+2}$ equivalent to $P$ such that every down step $D$ on the right of $C_{k+2}$ is not isolated.*

*Proof.* First, we apply Proposition 1 on $P$ which ensures that the height of the rightmost isolated $D$ step is at most one. After this, we distinguish three cases according to the position and the height of the rightmost isolated $D$-step.

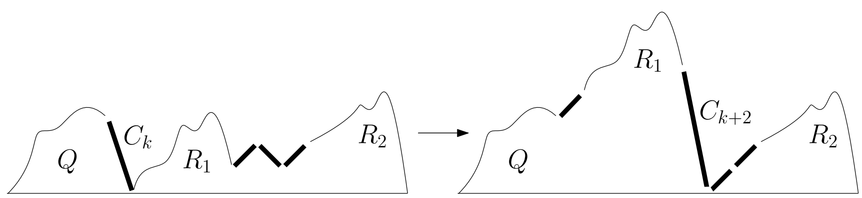
- if $R = R_1UD$, then the paths $QUR_1UC_{k+2}$ and $P$ are equivalent;



- if $R = R_1UDUR_2$ where $UR_2$ is a Dyck path, then $QUR_1UC_{k+2}UR_2$ and $P$ are equivalent;




- if $R = R_1UDUR_2$ where $UUR_2$ is a Dyck path then $QUR_1C_{k+2}UUR_2$ and $P$ are equivalent.




□

**Lemma 4** *Let $P$ be a path in $\mathcal{E}$ with only one catastrophe $C_k$, $k \geqslant 4$, satisfying Condition $(C)$. Then there is a path $Q$ with only one catastrophe $C_{k-2}$ equivalent to $P$, such that $C_k$ and $C_{k-2}$ are in the same position in $P$ and $Q$.*

*Proof.* Before describing the construction of $Q$, it is worth to notice the following fact. There is an occurrence of $UUU$ at height $k-3$ on the left of $C_k$ in $P$. Indeed, let us consider the rightmost $U$ at height $k-3$ on the left of $C_k$; clearly it necessarily precedes an step $U$ which constitutes an occurrence of $UU$ at height $k-3$. Moreover, since $P$ avoids $UUDU$, the two steps $UU$ are necessarily followed by another step $U$, and $P$ can be decomposed $P = P_1UUUP_2C_kP_3$ where $P_1$ is a prefix of Dyck path ending at height $k-3$, $UP_2D$ is a Dyck path and $P_3$ is a Dyck path. We complete the proof by setting $Q = P_1UDUP_2C_{k-2}P_3$. See Figure 3 for an illustration of this construction. □

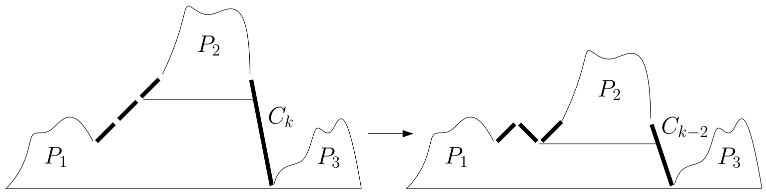



Figure 3: Illustration for Lemma 4.

### 3.7.1 The case of odd length

Let $\mathcal{A}'$ be the subset of paths in $\overline{\mathcal{A}}$ avoiding $UDU$, and not ending with $UD$, which also is the set of paths in $\overline{\mathcal{A}}$ without isolated $D$-steps. Let $\mathcal{A}_2$ be the set of paths satisfying Condition $(C)$ and having only one catastrophe $C_2$ at the end.

**Proposition 2** *A set of representatives of the $DD$-equivalence classes of odd length Dyck paths with catastrophes, is the set $\mathcal{R}_1$ of paths $R = AA'$ where $A \in \mathcal{A}_2$ and $A' \in \mathcal{A}'$.*

*Proof.* First, let us prove that for any path $P \in \mathcal{E}$, there is a path $Q \in \mathcal{R}_1$ lying in the same class. We obtain $Q$ by applying the following process: $(i)$ we apply Lemma 1; $(ii)$ we apply Lemma 3; $(iii)$ we apply $\psi$ extended to Dyck paths with only one catastrophe (see Proposition 1); $(iv)$ we apply Lemma 4 and Proposition 1, and we repeat it until the size of the catastrophe reaches 2 (thanks to Lemma 2). In the case where $UD$ precedes $C_2$, i.e. $Q = Q_1UDC_2Q_2$, we replace the path with $Q_1UUC_4Q_2$ (see Fig. 4) and we apply Lemma 4 and Proposition 1.

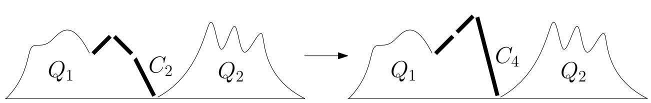



Figure 4: Illustration for Proposition 2.

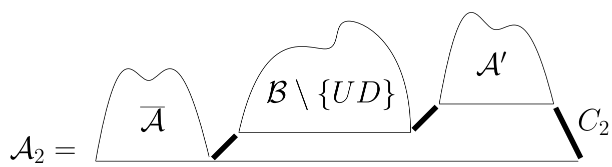
Now, it suffices to prove that two different paths $Q$ and $Q'$ in $\mathcal{R}_1$ with the same length cannot belong to the same class. Let us assume that $Q$ and $Q'$ lie in the same class. We decompose $Q = RC_2S$ and $Q' = R'C_2S'$. Suppose that $|S| \geqslant |S'|$. Since $S'$ does not contain isolated $D$-steps, $S'$ is entirely determined by the positions of occurrences of $DD$, which implies that $S$ has the form $S = TS'$. Since $S$ does not contain isolated $D$-step, $T$ cannot end with $UD$, and since $Q$ is $DD$-equivalent to $Q'$ it cannot end with $DD$, which means that $T$ is empty. So we have $|S| = |S'|$ and then $S = S'$. Using Proposition 1 for $RC_2$ and $R'C_2$, we obtain $Q = Q'$. □

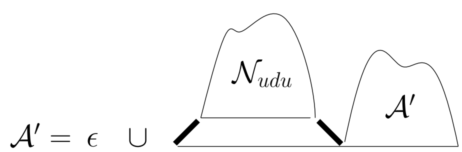
**Theorem 19** *(The case of odd length) The o.g.f. for the number of $DD$-equivalence classes of odd length paths in $\mathcal{E}$ is given by:*

$$N(x) = \frac{8x^3}{(x^2 + \sqrt{1 - 3x^4 - 2x^2} + 1)^2(-x^4 + (1 + x^2)\sqrt{1 - 3x^4 - 2x^2} - 2x^2 + 1)}.$$

*We have* $N(x) = x^3 + x^5 + 5x^7 + 11x^9 + 33x^{11} + 88x^{13} + 247x^{15} + O(x^{16})$.

*Proof.* By Proposition 2, it suffices to provide the o.g.f. $N(x)$ for $\mathcal{A}_2 \times \mathcal{A}'$ with respect to the length. Sets $\mathcal{A}_2$ and $\mathcal{A}'$ are constructed as follows




where $\mathcal{N}_{udu}$ is the set of non-empty Dyck paths avoiding $UDU$, $\overline{\mathcal{A}}$ is the set of representatives for the $DD$-equivalence classes of Dyck paths, e.g. Dyck paths avoiding $UUDU$ and having all $UDU$ at height 0 or 1, and $\mathcal{B}$ is the set of Dyck paths having all $UDU$ on $x$-axis and not starting with $UDU$. From [2] it follows that

$$\overline{A}(x) = \frac{1 - x^2 + \sqrt{-3x^4 - 2x^2 + 1}}{1 + x^6 + x^4 - 3x^2 - (x^4 - 1)\sqrt{-3x^4 - 2x^2 + 1}},$$

$$B(x) = \frac{2 - x^2 - x^4 + x^2\sqrt{-3x^4 - 2x^2 + 1}}{1 - x^2 + \sqrt{-3x^4 - 2x^2 + 1}},$$

and from [2, 13] we have $N_{udu}(x) = (x^2 - \sqrt{-3x^4 - 2x^2 + 1} + 1)/(2x^2) - 1$.

Finally, we compute $N(x) = \overline{A}(x)\,(B(x) - x^2)\,A'(x)x^3A'(x)$ where $A'(x)$ satisfies $A'(x) = 1 + x^2 N_{udu}(x)A'(x)$.

□

### 3.7.2 The case of even length

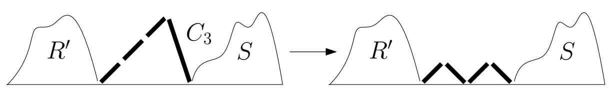
**Proposition 3** *A set of representatives of the DD-equivalence classes of even length Dyck paths with catastrophes is the set $\mathcal{R}_2$ defined by the union of $\overline{\mathcal{A}}$ with the sets $\mathcal{S}_0$ and $\mathcal{S}_1$ defined below:*

*- $\mathcal{S}_0$ is the set of Dyck paths $S = PDUUC_3$ where $PDU$ is a prefix of Dyck path satisfying condition $(C)$, and ending at height 2,*

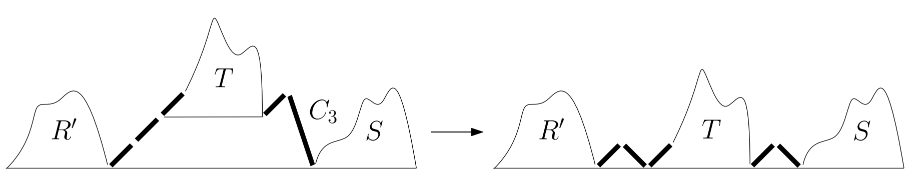
*- $\mathcal{S}_1$ is the set of Dyck paths $S = PDDC_3$ where $P$ is a prefix of Dyck path satisfying condition $(C)$, and ending at height 5.*

*Proof.* Let $P$ be a path in $\mathcal{E}$. If $P$ does not contain any catastrophe, then using [2], there is a unique path in $\overline{\mathcal{A}}$ equivalent to $P$. Now, let us assume that $P$ contains at least one catastrophe. We obtain $Q$ by applying the following process: $(i)$ we apply Lemma 1; we apply Lemma 3; we apply $\psi$ (extended to Dyck paths with only one catastrophe); we apply Lemma 4 followed by Proposition 1 until the path contains only one catastrophe $C_3$ (thanks to Lemma 2). We decompose the obtained path $Q = RC_3S$ where $S$ is a Dyck path with no isolated $D$-step. With the same argument used at the beginning of the proof of Lemma 4, $R$ is of the form $R = R'UUUR''$ where $R'$ and $UR''D$ are Dyck paths.

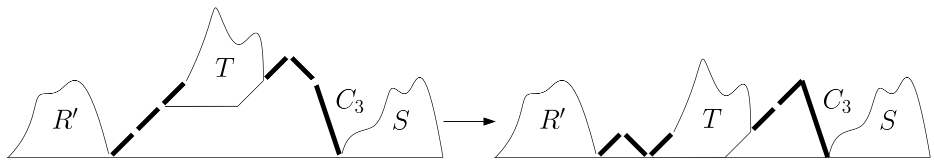
$(a)$ if $R''$ is empty, i.e. $R = R'UUU$, we replace $Q$ with $R'UDUDS \in \overline{\mathcal{A}}$;




$(b)$ if $R'' = TU$ ends with $U$, i.e. $R = R'UUUTU$, we replace $Q$ with $R'UDUTUDS \in \overline{\mathcal{A}}$;



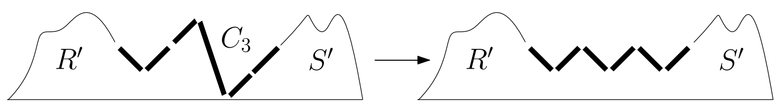
(c) if $R'' = TUD$ ends with $UD$, i.e. $R = R'UUUTUD$, we replace $Q$ with $R'UDUTUUC_3S$ (see the illustration below); In the case where $T$ ends with $U$, we apply the transformation (a) and we obtain a path in $\overline{\mathcal{A}}$;



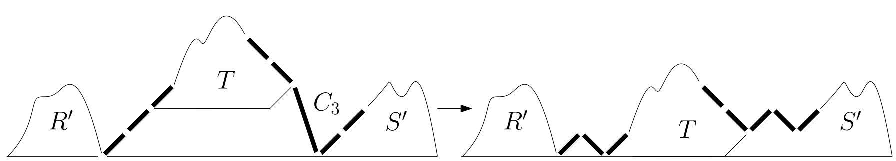

After applying all these transformations, the path is either in $\overline{\mathcal{A}}$, or it satisfies Condition $(C)$ and contains only one catastrophe $C_3$ that follows an occurrence $DUU$ or $DD$.

If $S$ is empty then the path lies in $\mathcal{R}_2$. Otherwise, $S$ does not contain isolated $D$-step and it starts necessarily with $UU$. In this case we set $S = UUS'$ and make the catastrophe disappear as follows:

(d) if $R$ ends at height 3 with $DUU$, we set $R = R'DUU$ and replace $Q$ with $R'DUDUDUS' \in \overline{\mathcal{A}}$;




(e) if $R$ ends with $UUUT$, where $UT$ is a prefix of Dyck path ending at height 1 and with $DD$, we set $R = R'UUUT$ and replace $Q$ by $R'UDUTUDUS'$.



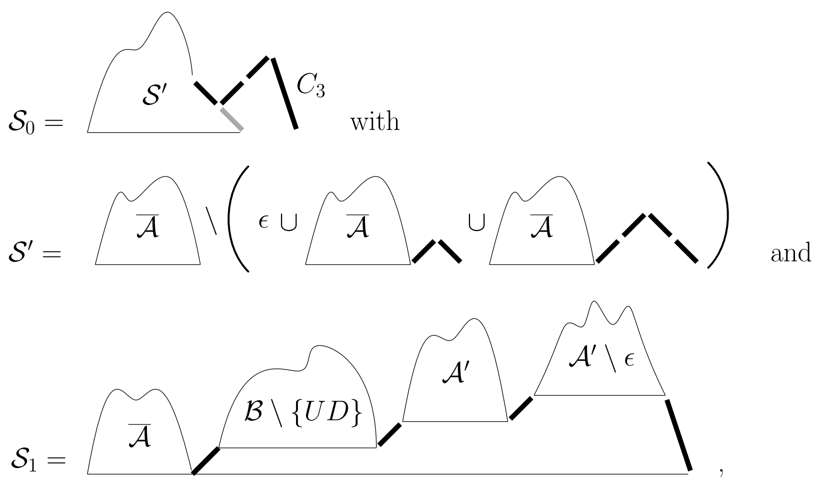

□

**Theorem 20** *(The case of even length) The o.g.f. $V(x)$ for the number of $DD$-equivalence classes of even length paths in $\mathcal{E}$ is given by:*

$$V(x) = \frac{4\left(x^4 - x^2 - 1\right)\sqrt{-3x^4 - 2x^2 + 1} + 8x^8 + 20x^6 + 8x^4 - 4}{\left(\left(-x^2 - 1\right)\sqrt{-3x^4 - 2x^2 + 1} + x^4 + 2x^2 - 1\right)\left(x^2 + \sqrt{-3x^4 - 2x^2 + 1} + 1\right)^2}.$$

*We have* $V(x) = 1 + x^2 + 2x^4 + 4x^6 + 11x^8 + 27x^{10} + 73x^{12} + 194x^{14} + 529x^{16} + 1448x^{18} + O(x^{19})$.

*Proof.* By Proposition 3, it suffices to provide the o.g.f. $M(x)$ for $\overline{\mathcal{A}} \cup \mathcal{S}_0 \cup \mathcal{S}_1$ with respect to the length. Sets $\mathcal{S}_0$ and $\mathcal{S}_1$ are constructed as follows:




where $\mathcal{A}'$ is the set of non-empty Dyck paths avoiding $UDU$ not ending with $UD$, $\overline{\mathcal{A}}$ is the set of representatives for the $DD$-equivalence classes of Dyck paths, e.g. Dyck paths avoiding $UUDU$ and having all $UDU$ at height 0 or 1, and $\mathcal{B}$ is the set of Dyck paths having all $UDU$ on $x$-axis and not starting with $UDU$.

Then, we have

$$S_0(x) = x^2 \cdot (\overline{A}(x) - 1 - x^2 \cdot \overline{A}(x) - x^4 \cdot \overline{A}(x))$$

and

$$S_1(x) = x^4 \cdot \overline{A}(x)(B(x) - x^2)A'(x)(A'(x) - 1)$$

which gives the claimed result after a simple calculation. □

### 3.7.3 The general case

Combining Theorem 19 and 20, we deduce:

**Theorem 21** *The o.g.f.* $L(x)$ *for the number of DD-equivalence in* $\mathcal{E}$ *is given by*

$$\frac{(4x^4 - 4x^2 - 4)\sqrt{-3x^4 - 2x^2 + 1} + 8x^8 + 20x^6 + 8x^4 - 8x^3 - 4}{\left((-x^2 - 1)\sqrt{-3x^4 - 2x^2 + 1} + x^4 + 2x^2 - 1\right)\left(x^2 + \sqrt{-3x^4 - 2x^2 + 1} + 1\right)^2},$$

*and it satisfies the following equation:* $x^{10}-x^9+4x^8-3x^7+5x^6-5x^5+2x^4-2x^3+x^2-x+1+(-x^{10}+x^9-6x^8+5x^7-9x^6+6x^5-3x^4+3x^3+x-1)\,L(x)+(x^{10}-x^9+4x^8-3x^7+5x^6-3x^5+2x^4-x^3)\,L(x)^2=0$.

*The first coefficients of* $x^n$*,* $n \geqslant 2$*, of the series expansion of* $L(x)$ *are* $1, 1, 2, 1, 4, 5, 11, 11, 27, 33, 73, 88, 194, 247, \ldots$*, and they do not correspond to a part of a sequence listed in [12].*

*An asymptotic approximation of these coefficients is*

$$(27.62956212 + 0.7009581600\,(-1)^n) \cdot \frac{\mathrm{e}^{0.5493061445 \cdot n}}{n^{\frac{3}{2}}}.$$

# References


[1] C. Banderier and M. Wallner, Lattice paths with catastrophes, *Electronic Notes in Discrete Mathematics*, **59** (2017), 131–146.

[2] J.-L. Baril and A. Petrossian, Equivalence of Dyck paths modulo some statistics, *Discrete Math.*, **338** (2015), 655–660.

[3] J.-L. Baril and A. Petrossian, Equivalence classes of Motzkin paths modulo a pattern of length at most two, *J. Integer Seq.*, **18** (2015), 15.7.1.

[4] J.-L. Baril and S. Kirgizov, Bijections from Dyck and Motzkin meanders with catastrophes to pattern avoiding Dyck paths, *Discrete Mathematics Letters*, **7** (2021), 5–10.

[5] J.-L. Baril, S. Kirgizov, and A. Petrossian, Enumeration of Łukasiewicz paths modulo some patterns, *Discrete Math.*, **342**(4) (2019), 997–1005.

[6] J.-L. Baril, J.L. Ramírez, L.M. Simbaqueba, Equivalence classes of skew Dyck paths Modulo some patterns, *Integers*, **22** (2022).

[7] E. Deutsch, Dyck path enumeration, *Discrete Math.*, **204**(1999), 167–202.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[9] A. Krinik, G. Rubino, D. Marcus, R.J. Swift, H. Kasfy, H. Lam. Dual processes to solve single server systems. *Journal of Stat. Planning and Inference*, 135(2005), 1, 121–147.

[10] K. Manes, A. Sapounakis, I. Tasoulas, and P. Tsikouras, Equivalence classes of ballot paths modulo strings of length 2 and 3, *Discrete Math.*, **339**(10) (2016), 2557–2572.

[11] A.G. Orlov, On asymptotic behavior of the Taylor coefficients of algebraic functions, *Siberian Mathematical Journal*, **25**(5) (1994), 1002–1013.

[12] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Available at https://oeis.org/.

[13] Y. Sun, The statistic "number of udu's" in Dyck paths, *Discrete Math.*, **287** (2004), 177–186.