$\mathbb{Q}$-bonacci words and numbers

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Abstract

We give a generalization of k-bonacci numbers where k is a positive rational number. These numbers enumerate binary words of size n whose maximal factors of the form 0^a1^b satisfy $ak > b$ or $a = 0$.

1 Introduction

Multi-step generalization of Fibonacci numbers, can be traced back to the works of Miles [12] and 14-year old Feinberg [6]. A lot of different studies about these numbers appear after, including the works of Flores [8], Miller [14], Dubeau [4] and Wolfram [17]. A bunch of combinatorial objects are enumerated by these numbers. For instance, the Knuth's exercise [11, p. 286] shows that the set of length n binary words avoiding k consecutive 1s is enumerated by k-bonacci numbers respecting $a_n = a_{n-1}+a_{n-2}+\cdots+a_{n-k}$, with initial conditions $a_0 = 1, a_{-1} = 1$, and $a_j = 0$ for any $j < -1$.

Independently, in two recent papers [1, 5], a new kind of restricted binary words enumerated by generalized Fibonacci numbers was considered. Baril, Kirgizov and Vajnovszki [1] defined a set $\mathcal{W}_{q,n}$, parameterized by a positive natural number q as follows:

Definition 1. $\mathcal{W}_{q,n}$ is the set binary words of length n such that for every maximal consecutive subword (factor) of the form 0^a1^b with $a > 0$ we have $aq > b$, where x^ℓ denotes a factor of length ℓ consisting only of symbols x. Figure 1 presents some examples.

Eğecioğlu and Iršič [5] deal with a graph R_n whose vertex set corresponds to the words from $\mathcal{W}_{1,n}$ starting with zero. Two vertices are adjacent in R_n if and only if the corresponding words differ at only one position.

We extend the above definition of $\mathcal{W}_{q,n}$ for the case where q is a positive rational number, provide generating functions and give a method to construct linear recurrence relation for the sequence $(|\mathcal{W}_{q,n}|)_{n\geqslant 0}$ with 0-or-1 coefficients.

0	00	000	0000
1	10	001	0001 1001
	11	100	0010 1100
		110	1000 1110
		111	1111

…

(a) $\mathcal{W}_{1,n}$ enumerated by Fibonacci.

0	00	000 100	0000 1000
1	01	001 101	0001 1001
	10	010 110	0010 1010
	11	111	0011 1100
			0100 1101
			0101 1110
			1111

…

(b) $\mathcal{W}_{2,n}$ enumerated by Tribonacci.

Figure 1: Sets $\mathcal{W}_{q,n}$ for small values of n and q.

2 Set construction and generating function

For $q \in \mathbb{Q}^+$, the set $\mathcal{W}_q = \bigcup_{n\in\mathbb{N}} \mathcal{W}_{q,n}$ is constructed as follows:

$$\mathcal{W}_q = \bigcup_{k=0}^{\infty}\{1^k\} \cup \mathcal{W}_q \cdot \mathcal{S}_q, \text{ where } \mathcal{S}_q = \bigcup_{i=0}^{\infty}\{\overbrace{0\dots00}^{1+\lfloor \frac{i}{q}\rfloor \text{ zeros}}\underbrace{1\dots11}_{i \text{ ones}}\}, \tag{1}$$

and $\mathcal{W}_q \cdot \mathcal{S}_q$ corresponds to a set of all possible concatenations of elements from $\mathcal{W}_q$ and $\mathcal{S}_q$ in this order. Table 1 shows shortest elements of $\mathcal{S}_q$ for different values of q. A word 111000010000110010 $\in \mathcal{W}_{1,18}$ decomposes as 111 0 0 001 0 00011 001 0, but a word 111000010000110010 $\in \mathcal{W}_{2,18}$ decomposes as 111 0 0 0 01 0 0 0011 0 01 0 and 111000010000110010 $\notin \mathcal{W}_{1/2}$ because the factor 001 is not in $\mathcal{S}_{1/2}$ and the word cannot be constructed.

$\mathcal{S}_{1/2}$	$\mathcal{S}_{2/3}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_{3/2}$
0	0	0	0	0
0001	001	001	01	01
0000011	000011	00011	0011	0011
0000000111	00000111	0000111	00111	000111
0000000001111	00000001111	000001111	0001111	0001111
0000000000011111	0000000011111	00000011111	00011111	000011111
…	…	…	…	…

Table 1: Shortest elements from sets $\mathcal{S}_q$.

Let $S_q(x) = \sum_{n=0}^{\infty} s_n x^n$ and $W_q(x) = \sum_{n=0}^{\infty} w_n x^n$ be the generating functions (g.f.) for $\mathcal{S}_q$ and $\mathcal{W}_q$, with respect to the word length, marked by x. Coefficients s_n and w_n are the numbers of words of length n from sets $\mathcal{S}_q$ and $\mathcal{W}_q$. Using the classical symbolic method to derive formulas for generating functions (see Flajolet-Sedgewick book [7]), we see that $\bigcup_{k=0}^{\infty}\{1^k\}$ has the generating function $\frac{1}{1-x}$, and Eq. (1) gives $W_q(x) = \frac{1}{1-x} + W_q(x)S_q(x)$, so

$$W_q(x) = \frac{1}{\big(1 - S_q(x)\big)(1-x)}. \tag{2}$$

In the following we consider a more refined (bivariate) version of generating functions with respect to the number of zeros and ones. We note, with a slight abuse of notation,

$$W_q(y,z) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty} w_{r,i} z^r y^i, \tag{3}$$

where $w_{r,i}$ is the number of words in $\mathcal{W}_q$ having exactly r zeros and i ones. Easy to see that $\mathcal{W}_q(x)$ is retrieved from $\mathcal{W}_q(y,z)$ by replacing both y and z by x, that is $\mathcal{W}_q(x) = \mathcal{W}_q(x,x)$. The bivariate g.f. $S_q(y,z)$ is defined in a similar way. The set $\bigcup_{k=0}^{\infty}\{1^k\}$ has the generating function $\frac{1}{1-y}$, and instead of Eq. (2) we write

$$W_q(y,z) = \frac{1}{\big(1 - S_q(y,z)\big)(1-y)}. \tag{4}$$

Now we show how to construct the set of suffixes $\mathcal{S}_q(y,z)$ and derive its generating function $S_q(y,z)$.

Definition 2. Let $q = \frac{c}{d}$ be a positive rational number represented by the irreducible fraction (e.g. $4 = \frac{4}{1}$), a word factor $0^d 1^c$ is called *spawning infix*. The generating function with respect to the number of zeros (marked by z) and the number of ones (marked by y) for the spawning infix $0^d 1^c$ is $z^d y^c$.

We

Definition 3. A polynomial

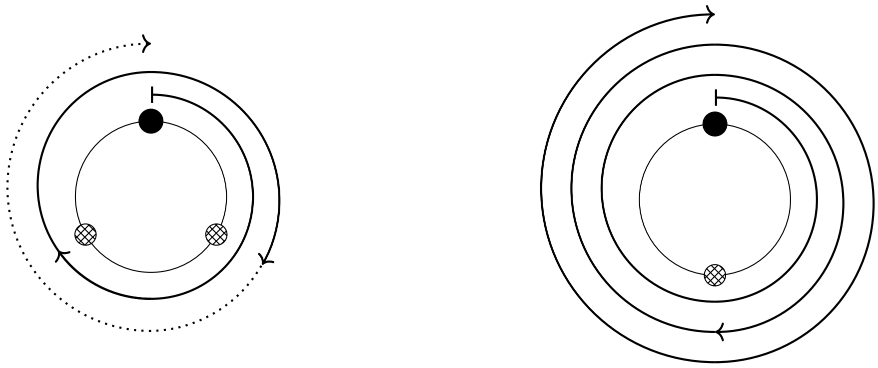
$$P_{q=\frac{c}{d}}(y,z) = \sum_{i=0}^{c-1} z^{1+\lfloor \frac{i}{q} \rfloor} y^i$$

is called *a model polynomial* of a positive rational number q represented by the irreducible fraction $q = \frac{c}{d}$.

We have, for instance, $P_{\frac{2}{3}}(y,z) = z + z^2 y$ and $P_{\frac{3}{2}}(y,z) = z + zy + z^2y^2$. It should be noted that the set of model polynomials is not in the bijection with the set of all positive rational numbers. The corresponding application is not injective, as we have $P_{1/k}(x) = z$ for any $k \in \mathbb{N}^+$. Figure 2 illustrates one of possible geometric interpretations of model polynomials and corresponding spawning infixes.

Lemma 1. *Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$. The generating function $S_q(y,z) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty} s_{r,i} z^r y^i$ where $s_{r,i}$ is the number of words of the form $0^r 1^i$, where $r = 1 + \lfloor i/q \rfloor$ is*

$$S_{q=\frac{c}{d}}(y,z) = \frac{P_q(y,z)}{1 - z^d y^c}.$$



(a) $P_{q=\frac{2}{3}}(y,z) = z + z^2 y$. The spawning infix 00011 corresponds to z^3y^2, represented by 3 arc-steps of the angle $4\pi/3$.

(b) $P_{q=\frac{3}{2}}(y,z) = z + zy + z^2y^2$. The spawning infix 00111 corresponds to z^2y^3, represented by 2 arc-steps of the angle 3π.

Figure 2: A graphical representation of two model polynomials and corresponding spawning infixes. For $j > 0$ a term $z^i y^j$ in a model polynomial corresponds to the suffix $0^i 1^j$. It means that one must make i arc-steps of the angle $2q\pi$ in order to strictly cross the starting point j times. The starting point is located at the north of the circle. The spawning infix is formed when we first time arrive exactly at the starting point.

Proof. Let us construct the set $\mathcal{S}_q$ in relation (1) iteratively. First add the word 0 and all words of the form $0^{1+\lfloor i/q \rfloor} 1^i$ for $i \in [1, c-1]$. These words correspond to the terms of the model polynomial $P_q(y,z)$. Other words of $\mathcal{S}_q$ are obtained by iteratively injecting the spawning infix $0^d 1^c$ just after the rightmost 0 in already generated words. Using the classical symbolic method [7] we see that $\frac{1}{1-z^d y^c}$ generates a sequence of infix additions. By construction $s_{r,i}$ is either 0 or 1. □

To illustrate Lemma 1 we take $q = 3/2$. In this case, the model polynomial is

$$P_{\frac{3}{2}}(y,z) = z + zy + z^2y^2$$

and corresponding words are

$$0, 01, 0011.$$

When $q = 3/2$ the spawning infix is 00111. Adding the infix just after the rightmost 0 we obtain

$$0\underline{00111}, 0\underline{00111}1, 00\underline{00111}11.$$

And repeating this operation, we get

$$000\underline{00111}111, 000\underline{00111}1111, 0000\underline{00111}11111, 00000\underline{00111}111111, \ldots$$

Finally, we obtain the set $\mathcal{S}_{\frac{3}{2}}$.

Theorem 1. *The generating function $W_q(y,z) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} w_{r,i} z^r y^i$ where $w_{r,i}$ is number of words from $\mathcal{W}_q$ of length $r+i$ containing exactly r zeros and i ones is*

$$W_q(y,z) = \frac{1 - z^d y^c}{(1-y)\big(1 - z^d y^c - P_q(y,z)\big)}.$$

Proof. It follows directly from Lemma 1 and Equation (4). □

Evaluating $W_q(x,x)$ we get the generating function $W_q(x) = \frac{1-x^{c+d}}{(1-x)\left(1-x^{c+d}-P_q(x,x)\right)}$ where x marks the length of words.

The total number of 0s (in other words, the *popularity* of 0s) in all words from $\mathcal{W}_{q=1,n}$ is enumerated by a shift of the sequence A6478 in Sloane's On-line Encyclopedia of Integer Sequences [15]. The corresponding g.f. is obtained by evaluating $\frac{\partial W_1(x,xz)}{\partial z}|_{z=1}$. It is not quite expected, but the sequence A6478 enumerates also the edges in the *Fibonacci hypercube* considered by Rispoli and Cosares [16]. A Fibonacci hypercube as a polytope determined by the convex hull of the *Fibonacci cube* which in turn is defined by Hsu in [10] as the graph whose vertices correspond to binary words of size n avoiding two consecutive 1s and where two vertices are connected if and only if the corresponding words differ at only one position. Is it possible to give some kind of a nice bijective construction between the edges of Fibonacci Hypercube and the 0s in words from $\mathcal{W}_{q=1,n}$? As far as we could check, no other sequences in OEIS [15] correspond to the popularity of 0s (or 1s) for other values of q.

3 Linear recurrence with 0-1 coefficients

We shall prove the following result.

Theorem 2. *Let a positive rational number q be represented by the irreducible fraction $\frac{c}{d}$. The number of n-length binary words from $\mathcal{W}_{q,n}$, denoted by w_n, can be expressed as*

$$w_n = \sum_{j\in J} w_{n-j} + w_{n-(c+d)}, \tag{5}$$

where J is the set of powers from the model polynomial $P_{q=\frac{c}{d}}(x,x)$. For example, when $q=\frac{3}{2}$, we have $P_{\frac{3}{2}}(x,x) = x + x^2 + x^4$, and $J = \{1,2,4\}$.

Initial conditions $w_0, w_1, \cdots, w_{c+d-1}$ are obtained by setting $w_n = 0$ for $n < 0$, unrolling Equation (5) from left to right, while adding an extra 1 for every w_i for $0 \leqslant i < c+d$.

Proof. Consider the following map ψ (first defined in [1]) acting on binary words

$$\psi(1^k) = 1^{k+c+d};$$
$$\psi(v1^\ell) = v0^d1^c1^\ell, \text{if } v \text{ ends with } 0.$$

We first show that ψ induces a bijection from $\mathcal{W}_{q,k}$ to the subset of words from $\mathcal{W}_{q,k+c+d}$ ending by at least c 1s. The map ψ inserts the spawning suffix 0^d1^c just after the rightmost 0 in a word having at least one 0. This does not change the property characterizing the words in $\mathcal{W}_q$ (see Definition 1). If there is no 0s in a word from $\mathcal{W}_{q,k}$, this word is extended by adding $c+d$ 1s at the end. And again it does not change the characterizing property of $\mathcal{W}_q$. Given the above analysis, it easy to see that ψ applied to any word in $\mathcal{W}_{q,n}$ gives us a word in $\mathcal{W}_{q,n+c+d}$ and this application is injective and surjective at the same time.

As follows from Equation (1), any word from $\mathcal{W}_{q,n}$ is either 1^n or have a form ps where $s = 0^{1+\lfloor i/q\rfloor}1^i$ is a word in $\mathcal{S}_q$ for certain $i \geqslant 0$, such that $n \geqslant 1 + \lfloor i/q\rfloor + i$ and $p \in \mathcal{W}_{q,n-(1+\lfloor i/q\rfloor+i)}$. When $n \geqslant c+d$ there are $c+1$ cases:

(case 1**)** The words of $\mathcal{W}_{q,n}$ ending with 0 are obtained by adding 0 at the right end of words from $\mathcal{W}_{q,n-1}$. This corresponds to the first term, z, of the model polynomial $P_{q=\frac{c}{d}}(y,z) = \sum_{i=0}^{c-1} z^{1+\lfloor i/q \rfloor} y^i$.

(case k**,** $1 < k < c$**)** The words of $\mathcal{W}_{q,n}$ ending with k 1s are obtained by adding the suffix $0^{1+\lfloor k/q \rfloor} 1^k$ at the right end of words from $\mathcal{W}_{q,n-(1+\lfloor k/q \rfloor + k)}$. This corresponds to the term $z^{1+\lfloor k/q \rfloor} y^k$ of the model polynomial $P_q(y,z)$.

(case $c+1$**)** The words of $\mathcal{W}_{q,n}$ ending with at least c 1s are obtained from the words of $\mathcal{W}_{q,n-(c+d)}$ by applying ψ.

Considering cardinalities of the sets, these $c+1$ cases give us the claimed recurrence relation (5). To construct initial conditions $\mathcal{W}_{q,0}, \mathcal{W}_{q,1}, \mathcal{W}_{q,2}, \dots \mathcal{W}_{q,c+d-1}$, we use the same process as in the previously considered cases, assuming that $\mathcal{W}_{q,m}$ contains no words for every $m < 0$, and adding an extra word 1^k into every set $\mathcal{W}_{q,k}$ with $0 \leqslant k < c+d$, so $\mathcal{W}_{q,0}$ contains only the empty word 1^0. □

Table 2 presents some sequences. Remark, that recurrence relations for sequences $(|\mathcal{W}_{q,n}|)_{n \geqslant 0}$ are equal to the recurrence relations for certain restricted integer compositions (ordered partitions). For some values of q the sequence $(|\mathcal{W}_{q,n}|)_{n \geqslant 0}$ corresponds exactly to a shift of a sequence enumerating restricted compositions (see $q = 1/5$ in Table 2). For other values of q the initial conditions differ from those of integer compositions. Consider, for instance, the case $q = 3/5$. The recurrence relation is $w_n = w_{n-1} + w_{n-3} + w_{n-6} + w_{n-8}$. The same recurrence holds for the sequence enumerating the compositions of $n \geqslant 2$ into 1s, 3s, 6s and 8s, but the initial conditions are different. The sequence of compositions starts with 1, 2, 3, 4, 7, 11, 17, 27, while the sequence $(|\mathcal{W}_{3/5,n}|)_{n \geqslant 0}$ begins with 1, 2, 3, 5, 8, 12, 19, 30.

4 Gray codes

A *k-Gray code*, named after Gray's work [9], for a set A of words of length n is an arrangement of all words of A in such a way that any two consecutive words differ at most in k positions. As follows from a result of [1] (which applies to the rational case also), a 3-Gray code exists for every $\mathcal{W}_{q,n}$ with $n \geqslant 0$ and any positive rational q.

For some values of q and n no 1-Gray code can exist, for example when $q = 2/3$ we have 12 words: 7 with odd number of 1s and 5 with even number of 1s. It easy to check that there is no 1-Gray in this case:

Odd number of 1s	Even number of 1s
00001	00000
00100	10010
00010	10001
10000	11000
11001	11110
11100	
11111	

q	Sequence	Recurrence relation	OEIS (with shifts)
1/5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, ...	$w_n = w_{n-1} + w_{n-6}$	Compositions (ordered partitions) of n into 1s and 6s. A5708
1/4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, ...	$w_n = w_{n-1} + w_{n-5}$	C. into 1s and 5s. A3520
1/3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, ...	$w_n = w_{n-1} + w_{n-4}$	C. into 1s and 4s. A3269
2/5	1, 2, 3, 4, 6, 9, 13, 18, 26, 38, 55, 79, ...	$w_n = w_{n-1} + w_{n-4} + w_{n-7}$	C. into 1s, 4s and 7s. Not in OEIS.
1/2	1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, ...	$w_n = w_{n-1} + w_{n-3}$	Narayana's cows, A930
3/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-6} + w_{n-8}$	NEW
2/3	1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5}$	C. into 1s, 3s and 5s, A60961
3/4	1, 2, 3, 5, 8, 13, 21, 33, 53, 85, 136, 218, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7}$	C. into 1s, 3s, 5s and 7s, A117760
4/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7} + w_{n-9}$	NEW
1	1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...	$w_n = w_{n-1} + w_{n-2}$	Fibonacci numbers, A45
5/4	1, 2, 4, 7, 13, 23, 42, 75, 136, 244, 441, 794, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-8} + w_{n-9}$	NEW
4/3	1, 2, 4, 7, 13, 23, 42, 75, 136, 245, 443, 799, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-7}$	NEW
3/2	1, 2, 4, 7, 13, 23, 42, 76, 138, 250, 453, 821, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5}$	NEW
5/3	1, 2, 4, 7, 13, 24, 44, 81, 148, 272, 499, 916, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5} + w_{n-7} + w_{n-8}$	NEW
2	1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3}$	Tribonacci numbers, A73
5/2	1, 2, 4, 8, 15, 29, 56, 107, 206, 396, 761, 1463, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-5} + w_{n-6} + w_{n-7}$	NEW
3	1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4}$	Tetranacci numbers, A78
4	1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5}$	Pentanacci numbers, A1591
5	1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5} + w_{n-6}$	Hexanacci numbers, A1592
...	...	...	...

Table 2: Cardinalities of $\mathcal{W}_{q,n\geqslant 0}$ for some values of q.

In general the question whether a 1-Gray code exists for a given q is a challenging one. Eğecioğlu-Iršič conjecture [5] is essentially about the existence of a 1-Gray code for $W_{1,n}, n \geqslant 0$. In [1] this conjecture was proved by presenting a sophisticated recursive construction. Here is an example for the words of length 5 and $q = 1$: 11111, 11110, 11100, 11000, 11001, 10001, 10000, 10010, 00010, 00011, 00001, 00000, 00100. Moreover, experimental investigations for small values, $0 \leqslant n \leqslant 5$ and $2 \leqslant q \leqslant 5$, suggest the following conjecture: a 1-Gray code exist for $\mathcal{W}_{q,n}$ where $q \in \mathbb{N}^+$ for any $n \geqslant 0$.

5 Generalized golden ratio

The generalized golden ratio is defined as $\varphi_k = \lim_{n\to\infty} a_{n+1}/a_n$, where a_{n+1} and a_n are two adjacent k-bonacci numbers. The golden ratio is $\varphi_2 = (1+\sqrt{5})/2$, and $\varphi_3 = (1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}})/3$ is known as the Tribonacci constant. The Tetranacci constant φ_4 have quite a large expression in radicals. In general, φ_k is expressed as the largest root of the polynomial $x^k - x^{k-1} - \cdots - x - 1$. See Wolfram's paper [17] for full details. In the same paper Wolfram conjectured that there is no expression in radicals for $k \geqslant 5$. By computing the Galois group, with the help of the computer algebra system Magma [2], he confirmed the conjecture for $5 \leqslant k \leqslant 11$. Martin [13] proved the case of even or prime k. Furthermore, Cipu and Luca [3] demonstrated the impossibility of the construction of φ_k by ruler and compass for $k \geqslant 3$. As far as we can tell, the question whether there is an expression in radicals remains open for prime $k > 11$. Dubeau [4] proved that φ_k approaches 2 when $k \to \infty$.

Our generalization of k-step Fibonacci numbers allows to use any positive rational value of k and construct a corresponding set restricted binary words of length n, denoted in this paper by $\mathcal{W}_{q,n}$ with $q = k + 1$. Actually the set is well-defined even if we extend the domain of the parameter q to all positive real numbers. Can we prove that the ratio $\frac{|\mathcal{W}_{r,n+1}|}{|\mathcal{W}_{r,n}|}$ have a limit for every $r \in \mathbb{R}$? If so, can we also prove that that a function associating to any positive real value the limit of the aforementioned ratio is continuous?

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