Asymptotics of strongly overlapping permutations

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Abstract

In this work, we introduce the concept of strongly (non-)overlapping permutations, which is related to the larger study of consecutive patterns in permutations. We show that this concept admits a simple and clear geometrical meaning, and prove that a permutation can be represented as a sequence of non-overlapping ones. The above structural decomposition allows us to obtain equations for the corresponding generating functions, as well as the complete asymptotic expansions for the probability that a large random permutation is strongly (non-)overlapping. In particular, we show that almost all permutations are strongly non-overlapping, and that the corresponding asymptotic expansion has the self-reference property: the involved coefficients count strongly non-overlapping permutations once again. We also discuss the similarities of the introduced concept to already existing permutation building blocks, such as indecomposable and simple permutations, as well as their associated asymptotics.

Keywords: asymptotics, permutations, consecutive patterns, overlapping.

1 Introduction

The study of *pattern-avoiding permutations* is a dynamically developing area of mathematics that has been exciting scientists since the mid-1980s. Since the first systematic enumeration by Simion and Schmidt [17], interest in permutation patterns has grown steadily, which can be explained by the presence of extensive connections with other mathematical fields. One of the important variations of this study concerns the notion of *consecutive patterns*, first systematically treated by Elizalde and Noy [11]. To obtain generating functions and distributions of pattern occurrences in various combinatorial objects, one can employ powerful Goulden–Jackson *cluster method* [12, 13, 14]. As a particular example, we would like to cite the proofs of Elizalde–Noy and Nakamure conjectures [10].

Understanding the overlaps of combinatorial sub-structures is of core importance in the "cluster" method. This need is reflected in the concept of non-overlapping permutations, also known as minimally overlapping [5, 9, 16]: a permutation of length n is *non-overlapping* if it does not contain isomorphic prefix and suffix of size $k \in \{2, \dots, n-1\}$. Here, we call two sequences of distinct integers $\pi = \pi_1\pi_2 \cdots \pi_k$ and $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ *isomorphic* if $\pi_i > \pi_j \Leftrightarrow \sigma_i > \sigma_j$ for all possible i and j. For instance, permutation 132 is non-overlapping, but 1324 is overlapping, since its prefix 13 is isomorphic to its suffix 24. The counting sequence of non-overlapping permutations starts with

$$1,\ 2,\ 4,\ 12,\ 48,\ 280,\ 1\,864,\ 14\,840,\ 132\,276,\ 1\,323\,504,\ \dots,$$

see A263867 entry in Sloane's Encyclopedia [18].

Motivated by the ongoing study [4] related to distributions of special kind of patterns in RNA secondary structures with allowed pseudoknots, modelled as fixed-point free involutions, in this paper we introduce a notion of *strongly overlapping* permutations, which is similar yet different from Bóna's overlapping permutations.

Definition 1.1. *A permutation $\sigma \in S_n$ is* strongly overlapping, *if there is an integer $1 \le k < n$ called* overlapping range *such that the following three conditions hold:*

1. *the interval $\{1, \dots, k\}$ is invariant under σ,*

2. *the interval $\{n-k+1, \dots, n\}$ is invariant under σ,*

3. *the first and last k consecutive positions of σ form isomorphic patterns.*

Otherwise, we call σ strongly non-overlapping. *We designate by $\mathfrak{s}_n$ and $\mathfrak{n}_n$, respectively, the numbers of strongly overlapping and strongly non-overlapping permutations of size n.*

From a geometric point of view, if a permutation $\sigma \in S_n$ is strongly overlapping with an overlapping range k, then its plot possesses two congruent blocks of size $k \times k$: one of them is located at the lower left corner, while the other is in the upper right one. For example, the permutation $\sigma = 214365$ is strongly overlapping with overlapping ranges $k = 2$ and $k = 4$ (Figure 1).

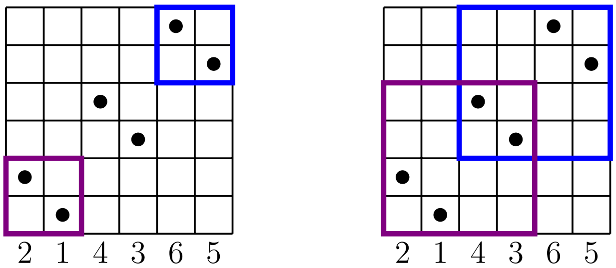

Figure 1: Overlapping blocks in the permutation 214365.

Figure 2 illustrates all strongly overlapping permutation of size $n \le 4$. Observing these and other examples, the reader may give a guess that a strongly overlapping permutation possesses an overlapping range of no more than half its size. In Section 2, we show that this is, indeed, the case (Lemma 2.1). This observation gives rise to a structural decomposition of permutations into direct sums of strongly non-overlapping ones (Theorem 2.4). In its turn, the decomposition leads to relations on generating functions (Theorem 2.5), that allows us to establish counting sequences $(\mathfrak{s}_n)$ and $(\mathfrak{n}_n)$.

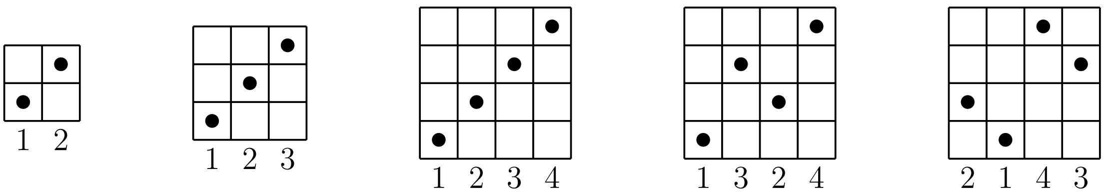

Figure 2: Strongly overlapping permutations of size at most 4.

The first few values of the sequence $(\mathfrak{s}_n)$ are

$$\mathfrak{s}_n = 0,\ 1,\ 1,\ 3,\ 7,\ 31,\ 131,\ 775,\ 5\,211,\ 41\,315,\ \dots$$

while the complementary sequence $(\mathfrak{n}_n)$ begins with

$$\mathfrak{n}_n = 1,\ 1,\ 5,\ 21,\ 113,\ 689,\ 4\,909,\ 39\,545,\ 357\,669,\ 3\,587\,485,\ \dots$$

A single glance at these numbers is enough to conjecture that with high probability a typical large permutation is strongly non-overlapping. We prove this conjecture at Section 3. Moreover, we establish a complete asymptotic expansion of the probability that a uniform random permutation is strongly overlapping (Theorem 3.1). It turns out that the coefficients involved into the expansion admit combinatorial interpretation: they are $\mathfrak{n}_n$ again. Furthermore, the expansion under consideration resembles another one related to indecomposable permutations. This observation raises several questions that are discussed in Section 4.

2 Structure and enumeration

Lemma 2.1. *Any strongly overlapping permutation of size n admits an overlapping range of size at most $n/2$.*

Proof. Suppose that an overlapping range k of a strongly overlapping permutation $\sigma \in S_n$ is greater than $n/2$. In this case, the lower left and upper right blocks of size $k \times k$ are congruent and have non-empty intersection. The intersection is a $(2k-n)\times(2k-n)$ block, which is congruent to the lower left and upper right blocks of the same size. Therefore, σ admits overlapping range $(2k-n) < k$. If $(2k-n) < n/2$, we are done. Otherwise, repeat the procedure until the overlapping block sizes are less than $n/2$ (Figure 3). □

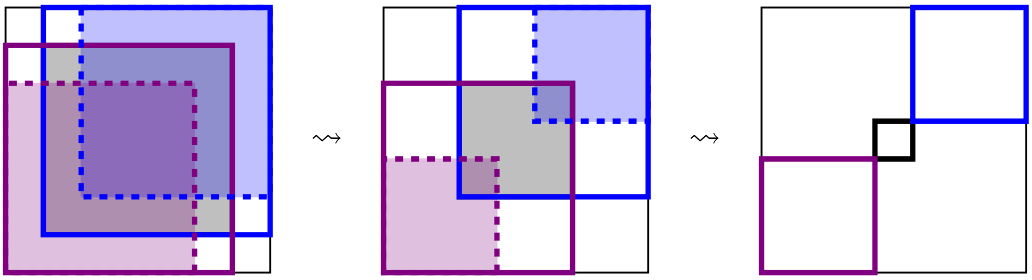

Figure 3: Schema for reducing an overlapping range. If overlapping blocks have a non-empty intersection (gray areas on the figure), then this intersection can be slided to get smaller overlapping blocks (blue and violet areas).

Recall that the *direct sum* of permutations $\pi \in S_m$ and $\tau \in S_n$ is the permutation $\pi \oplus \tau$ of length $(m+n)$ defined by

$$(\pi \oplus \tau)(i) = \begin{cases} \pi(i) & \text{for } 1 \leq i \leq m, \\ \tau(i-m)+m & \text{for } m+1 \leq i \leq n. \end{cases}$$

Lemma 2.1 allows us to decompose strongly overlapping permutations into a direct sum of smaller permutations (schematically, the summands are represented by consecutive non-intersecting blocks, see the right part of Figure 3). More precisely, we get the following structural result.

Lemma 2.2. *Any strongly overlapping permutation σ can be uniquely decomposed into a direct sum*

$$\sigma = \pi \oplus \tau \oplus \pi,$$

where π is strongly non-overlapping permutation and τ is arbitrary (possibly, empty) permutation.

Proof. Let k be the minimal overlapping range of σ. According Lemma 2.1, $k \leq n/2$. Hence, σ possesses three consecutive invariant intervals: $\{1,\ldots,k\}$, $\{k+1,\ldots,n-k\}$ and $\{n-k+1,\ldots,n\}$ (it may happen that the second of them is empty). This gives us the above decomposition $\sigma = \pi\oplus\tau\oplus\pi$. Note that the permutation $\pi \in S_k$ is strongly non-overlapping due to minimality of the overlapping range k. Indeed, if π is strongly overlapping with an overlapping range l, then $l < k$ is the overlapping range of σ too, which leads to a contradiction (see Figure 4). □

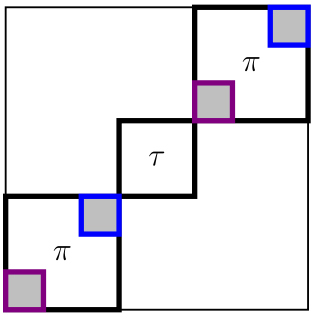

Figure 4: Structure of strongly overlapping permutations. If the overlapping block π is a strongly overlapping permutation itself, then its size is not the minimal overlapping range of the permutation $\sigma = \pi \oplus \tau \oplus \pi$.

Corollary 2.3. *The counting sequence* $(\mathfrak{s}_n)$ *of strongly overlapping permutations satisfies*

$$\mathfrak{s}_n = \sum_{k=1}^{[n/2]} \mathfrak{n}_k \cdot (n-2k)!.$$

Theorem 2.4. *Any permutation* σ *can be uniquely decomposed into a direct sum of strongly non-overlapping permutations,*

$$\sigma = \pi_1 \oplus \ldots \oplus \pi_m \oplus \tau \oplus \pi_m \oplus \ldots \oplus \pi_1, \tag{1}$$

where the permutation τ *is, possibly, empty.*

Proof. If σ is strongly non-overlapping, then decomposition (1) is done with $m = 0$ and $\tau = \sigma$. Otherwise, let us apply Lemma 2.2 iteratively. In other words, express σ as $\pi_1 \oplus \tau_1 \oplus \pi_1$. If τ_1 is strongly overlapping, then express it as $\pi_2 \oplus \tau_2 \oplus \pi_2$, etc. Thus, after a finite number of iterations, we obtain decomposition (1).

Now, let us show the uniqueness of the above decomposition. Suppose the contrary, that is, suppose that we have two different decompositions of form (1):

$$\pi_1 \oplus \ldots \oplus \pi_m \oplus \tau \oplus \pi_m \oplus \ldots \oplus \pi_1 = \pi'_1 \oplus \ldots \oplus \pi'_{m'} \oplus \tau' \oplus \pi'_{m'} \oplus \ldots \oplus \pi'_1.$$

Without loss of generality, we can assume that $\pi_1 \neq \pi'_1$, which means that they are of different sizes, say, l and l'. However, if $l < l'$, then π'_1 is strongly overlapping with the overlapping range l (see Figure 5). The case $l' < l$ leads to a similar contradiction. Thus, $\pi_1 = \pi'_1$, and decomposition (1) is unique. □

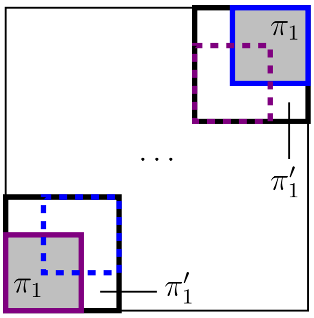

Figure 5: Uniqueness of decomposition (1).

Let us consider the generating functions $S(z)$ and $N(z)$ of strongly overlapping and strongly non-overlapping permutations, respectively:

$$S(z) = \sum_{n=1}^{\infty} \mathfrak{s}_n z^n \qquad \text{and} \qquad N(z) = \sum_{n=1}^{\infty} \mathfrak{n}_n z^n.$$

We formally assume that an empty permutation is neither strongly overlapping nor strongly non-overlapping. Hence, the generating function $P(z)$ of all permutations can be represented as the following sum:

$$P(z) = 1 + S(z) + N(z) = \sum_{n=0}^{\infty} n!\, z^n. \tag{2}$$

This observation, together with Lemma 2.2, helps us establish relations that determine the behavior of $S(z)$ and $N(z)$.

Theorem 2.5. *The generating functions $S(z)$ and $N(z)$ satisfy the following relations:*

$$N(z) = P(z)\big(1 - N(z^2)\big) - 1 \tag{3}$$

and

$$S(z) = \frac{1 + N(z)}{1 - N(z^2)} \cdot N(z^2). \tag{4}$$

Proof. In terms of generating functions, Lemma 2.2 can be interpreted as the following relation:

$$S(z) = P(z) \cdot N(z^2).$$

At the same time, it follows from (2) that

$$S(z) = P(z) - N(z) - 1.$$

Equating the right-hand sides of these two identities, we obtain relation (3). Meanwhile, to get (4), it is sufficient to take the first of the identities and replace $P(z)$ with the expression obtained via relation (3). □

Remark 2.6. *Relation* (3) *rewritten as*

$$P(z) = \frac{1 + N(z)}{1 - N(z^2)} \tag{5}$$

is another face of Theorem 2.4. Indeed, the factor $1 + N(z)$ corresponds to the central block of decomposition (1)*, while $\big(1 - N(z^2)\big)^{-1}$ represents the sequence of strongly non-overlapping permutations. Note that the argument z^2 reflects the fact that each element of the sequence is taken twice. Relation* (4) *carries a similar meaning. The only difference is the presence of the additional factor $N(z^2)$ that guarantees that the decomposition consists of at least two blocks.*

3 Asymptotics

Theorem 3.1. *For any positive integer r, the probability that a uniform random permutation $\sigma \in S_n$ is strongly overlapping, as $n \to \infty$, satisfies*

$$\mathbb{P}(\sigma \textit{ is strongly overlapping}) = \sum_{k=1}^{r-1} \frac{\mathfrak{n}_k}{(n)_{2k}} + O\left(\frac{1}{n^{2r}}\right), \tag{6}$$

where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials.

Proof. It follows from Corollary 2.3, that the counting sequence of strongly overlapping permutations satisfies

$$\mathfrak{s}_n \leq \sum_{k=1}^{r-1} \mathfrak{n}_k \cdot (n-2k)! + \sum_{k=r}^{[n/2]} k! \cdot (n-2k)!.$$

Let us estimate the second summand. For n large enough and $r < k \leq n/2$, we have

$$k! \cdot (n-2k)! \leq (r+1)! \cdot (n-2r-2)!.$$

As a consequence,

$$\sum_{k=r}^{[n/2]} k! \cdot (n-2k)! \leq r! \cdot (n-2r)! + (n-r) \cdot (r+1)! \cdot (n-2r-2)! = O\big((n-2r)!\big),$$

which leads us to

$$\mathfrak{s}_n = \sum_{k=1}^{r-1} \mathfrak{n}_k \cdot (n-2k)! + O\big((n-2r)!\big).$$

Now, since $\mathbb{P}(\sigma \text{ is strongly overlapping}) = \mathfrak{s}_n/n!$, it is sufficient to divide the obtained inequality by $n!$. □

Let $\mathfrak{s}_n^{(m)}$ be the number of permutations of size n whose decomposition (1) consists of $(2m+1)$ summands (in particular, $\mathfrak{s}_n^{(0)} = \mathfrak{n}_n$). Define also $\mathfrak{n}_n^{(m)}$ to be the number of permutations that can be represented as a direct sum of m strongly non-overlapping permutations (in particular, $\mathfrak{n}_0^{(0)} = 1$, $\mathfrak{n}_n^{(0)} = 0$ for $n > 0$, and $\mathfrak{n}_n^{(1)} = \mathfrak{n}_n$).

Theorem 3.2. *For any positive integer r, the probability that decomposition* (1) *of a uniform random permutation $\sigma \in S_n$ consists of* $(2m+1)$ *summands, as $n \to \infty$, satisfies*

$$\mathbb{P}(\sigma \textit{ consists of } 2m+1 \textit{ blocks}) = \sum_{k=1}^{r-1} \frac{\mathfrak{n}_k^{(m)} - \mathfrak{n}_k^{(m+1)}}{(n)_{2k}} + O\left(\frac{1}{n^{2r}}\right), \qquad (7)$$

where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials.

Proof. Due to Lemma 2.2 and Theorem 2.4, the number of permutations whose decomposition (1) consists of at least $(2m+1)$ summands is equal to

$$\sum_{k=1}^{[n/2]} \mathfrak{n}_k^{(m)} \cdot (n-2k)!.$$

Therefore, the sequence $\big(\mathfrak{s}_n^{(m)}\big)$ satisfies

$$\mathfrak{s}_n^{(m)} = \sum_{k=1}^{[n/2]} \left(\mathfrak{n}_k^{(m)} - \mathfrak{n}_k^{(m+1)}\right) \cdot (n-2k)!.$$

Similarly to the previous theorem, for any tail of this sum, we have an estimation

$$\sum_{k=r}^{[n/2]} \left(\mathfrak{n}_k^{(m)} - \mathfrak{n}_k^{(m+1)}\right) \cdot (n-2k)! = O\big((n-2r)!\big).$$

Thus, dividing the above relation by $n!$, we obtain asymptotic expression (7). □

4 Conclusion

As we have seen in Section 2, each permutation can be represented as a direct sum of strongly non-overlapping ones. Therefore, we can consider non-overlapping permutations as a sort of basic irreducible blocks that can serve for constructing the whole class of permutations. There are other examples of this kind in the literature related to permutations. One of them concerns so called *indecomposable permutations*, that is, permutations with no proper invariant interval of the form $\{1, \ldots, k\}$ (see, for example, the paper of Cori [8]). It is known that each permutation can be represented as a sequence of indecomposable ones. In particular, the generating function $I(z)$ of indecomposable permutations satisfies

$$I(z) = 1 - \frac{1}{P(z)}.$$

In the class of permutations, the strongly non-overlapping ones form the majority. In other words, almost all permutations are strongly non-overlapping. The same is true for indecomposable permutations. Moreover, this analogy can be extended to complete asymptotic expansions. It was Comtet [7] who first studied the probability that a large permutation is indecomposable, and established its expansion over the basis $1/(n)_k$. Later, it turned out that the involved coefficients have combinatorial meaning. More precisely, the following result was recently shown [15]: for any positive integer r, the probability that a uniform random permutation $\sigma \in S_n$ is indecomposable satisfies

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) = 1 - \sum_{k=1}^{r-1} \frac{2\mathfrak{i}_k - \mathfrak{i}_k^{(2)}}{(n)_k} + O\left(\frac{1}{n^r}\right), \tag{8}$$

where $(\mathfrak{i}_k)$ is the counting sequence of indecomposable permutations, and $(\mathfrak{i}_k^{(2)})$ counts permutations with exactly two indecomposable parts. From Theorem 3.1, we can see that the asymptotic expansion of the probability that a uniform random permutation is strongly non-overlapping has the same spirit:

$$\mathbb{P}(\sigma \text{ is strongly non-overlapping}) = 1 - \sum_{k=1}^{r-1} \frac{\mathfrak{n}_k}{(n)_{2k}} + O\left(\frac{1}{n^{2r}}\right). \tag{9}$$

Another example of irreducibilities involves *simple permutations* that do not map non-trivial intervals onto intervals (see [2]). The generating function $M(z)$ of simple permutations of size at least 4 can be found iteratively from the equation

$$\frac{P(z) - P^2(z)}{1 + P(z)} = z + M\big(P(z)\big).$$

The asymptotic expansion of the probability that a uniform random permutation $\sigma \in S_n$ is simple was established by Borinsky [6]:

$$\mathbb{P}(\sigma \text{ is simple}) = \frac{1}{e^2}\left(1 - \frac{4}{n} + \frac{2}{(n)_2} - \frac{40}{3(n)_3} - \frac{182}{3(n)_4} - \ldots\right). \tag{10}$$

Again, we observe the role of the basis $1/(n)_k$. However, compared to the cases related to strongly non-overlapping and indecomposable permutations, this expansion has two important differences. First, the leading term is not 1 anymore: the proportion of simple permutations tends to e^{-2}. Second, the involved coefficients are not integers anymore.

These observations raise a number of questions. First of all, it would be of interest to unite the three above discussed permutation classes under the same theory that could explain both the similarities and differences. A priori, this question is quite complicated. It would be natural to try employing the *wreath product* introduced by Atkinson and Stitt [3] for studying restricted permutations. Indeed, constructing the class of permutations from the simple ones admits description in terms of the wreath product [1]; that is true for indecomposable permutations as well. However, it looks like the strongly non-overlapping permutations do not allow such a description.

Another question is the following: is it possible to naturally modify the basis in asymptotics (10), so that the coefficients become integers? At first glance, it seems that it could be sufficient to divide the basis vectors $1/(n)_k$ by $k!$. This change suggests switching from the ordinary generating functions to the exponential ones. That trick works well for establishing complete asymptotic expansions of indecomposable permutations and indecomposable perfect matchings, see [15, Chapter 9]. The potential of its applicability for simple permutations needs to be verified.

Finally, if we get a positive answer to the previous question, what is the combinatorial interpretation of these coefficients? Unfortunately, there is no clue in the OEIS [18]. The only thing one can be sure of is that the coefficients cannot be interpreted as a counting sequence of combinatorial class: some of them are negative. Thus, we would rather expect linear combinations of different counting sequences, as is the case for indecomposable permutations (8).

5 Acknowledgements

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