Complexity of a “weak” orientation of a graph

Sergey . Kirgizov @ LIP6.fr
Julien . Bensmail @ LABRI.fr
Romaric . Duvignau @ LABRI.fr

June 25, 2013

Abstract

We say that a graph admits a weak orientation if there is a directed path joining any two vertices in either direction. We show that the problem of deciding whether a graph admits such orientation is P-complete.

Contents

0 Some notation

- *Digraph* is a directed graph.
- *Distance*, $dist(u, v)$, is the minimal length of a (di)path joining u and v.
- *Diameter of a graph* is the greatest distance between any pair of its vertices.
- *Strong diameter of a digraph* is the greatest distance between any pair of its vertices.
- *Weak (one-way) distance*, $dist_w(u, v)$ is equal to the length of the shortest dipath between u and v regardless of their order, i.e.

$$dist_w = \min\left(|dipath(u, v)|, |dipath(v, u)|\right).$$

- *Weak diameter of a digraph* is the greatest weak distance between any pair of vertices.
- An orientation of graph is *k-weak* (resp. *k-strong*) if it has weak (resp. strong) diameter at most k.
- An orientation of graph is *weak* (resp. *strong*) if it has finite weak (resp. strong) diameter.

1 Background and introduction

Let us introduce four decision problems and discuss already known their solutions. All of these problems consider the possibility of *good* orientation of a graph, where the notion of goodness slightly varies from problem to problem.

ORIENTATION WITH STRONG DIAMETER – OSD
Instance: A graph G.
Question: Does G admit a strong orientation?
Answer: This problem can be solved in linear time. Firstly, Robbins proved a theorem: **graph G admits strong orientation if and only if G is bridgeless** [5]. Secondly, thanks to the work of Tarjan and Hopcroft, we have a linear time algorithm to find all bridges [6]. And if there are no bridges, one can find a good orientation of G, using another algorithm [4, 1] developed by the same researchers.

ORIENTATION WITH STRONG DIAMETER k – k-OSD
Instance: A graph G.
Question: Does G admit a k-strong orientation?
Partial answer: The 1-OSD orientation is impossible. Chvátal and Thomassen proved that the 2-OSD problem is NP-complete [3].

ORIENTATION WITH WEAK DIAMETER – OWD
Instance: A graph G.
Question: Does G admit a weak orientation?

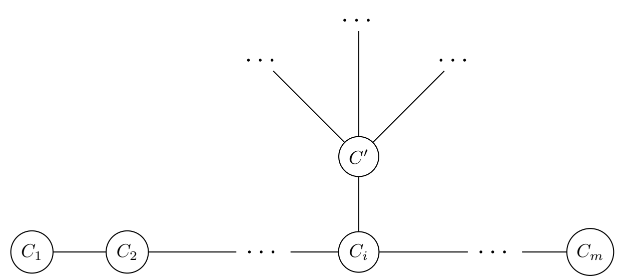
ORIENTATION WITH WEAK DIAMETER k – k-OWD
Instance: A graph G.
Question: Does G admit a k-weak orientation?
Answer: This problem is very simple when $k = 1$. Bensmail *et al.* showed that this problem is NP-complete for any $k \geq 2$ [2].

In the rest of the paper we will show that OWD can be solved in linear time.

2 Orientation with finite weak diameter

A graph G has a finite number bridges, and we can use a linear time algorithm [6] to find all of them. Removing all bridges from G one obtain a set of bridgeless components $\{C_i\}$. By Robbins' theorem each C_i admits a strong orientation. Particularly, it means that for every three vertices u, v and c from C_i there is a dipath from u to v passing through c and vice-versa.

A *B-contraction* of a graph consists in replacing all its bridgeless components by points. Note, that the result of B-contraction of a connected graph is a tree.



Theorem. *G admit a weak orientation if and only if the result of B-contraction is a path.*

Proof. We consider only connected graphs, because a disconnected one obviously does not admit *good* orientation.

$\Leftarrow$ If B-contraction results in a path, we construct a weak orientation of G as follows: (i) we perform strong orientation on each C_i; (ii) we orient all bridges in the same direction. That is, we go from path's start to its end and we orient bridges forward.

$\Rightarrow$ Given a weak orientation of G, we suppose that B-contraction does not results in a path. It means that contracted version of G contains a claw (a star, $K_{1,3}$) as an induced subgraph. But no one can find a weak orientation of a claw. Thus, by contradiction, B-contraction of weak-orientable graph G must have a path as a result.

$\square$

Note that the finding of all the edges, using Tarjan's algorithm takes linear time. Testing whether B-contraction gives us a path also very easy task. Finally, a strong orientation of a given bridgeless undirected graph may be found in linear time also.

References

[1] *Robbins' theorem. algorithms and complexity - wikipedia, the free encyclopedia.* http://en.wikipedia.org/wiki/Robbins%%27_theorem#Algorithms_and_complexity.

[2] J. Bensmail, R. Duvignau, and S. Kirgizov, *The complexity of deciding whether a graph admits an orientation with fixed weak diameter.*

[3] V. Chvátal and C. Thomassen, *Distances in orientations of graphs*, J. Combin. Theory, Ser. B, 24 (1978), pp. 61–75.

[4] J. Hopcroft and R. Tarjan, *Algorithm 447: efficient algorithms for graph manipulation*, Commun. ACM, 16 (1973), pp. 372–378.

[5] H. E. Robbins, *A theorem on graphs, with an application to a problem of traffic control*, The American Mathematical Monthly, 46 (1939), pp. 281–283.

[6] R. Tarjan, *A note on finding the bridges of a graph*, Information Processing Letters, 2 (1974), pp. 160 – 161.