Patterns in treeshelves

Jean-Luc Baril, Sergey Kirgizov, Vincent Vajnovszki

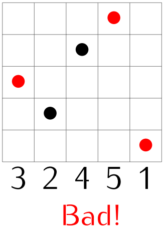
Université Bourgogne Franche-Comté

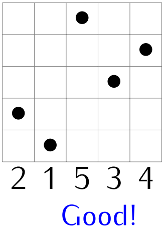
Séminaire Le2i CRSD
(Combinatoire, Réseaux et Sciences des Données)
14 Février, Dijon 2017

Knuth (1968) example

$4312 \xmapsto{\textit{sorting}} 1234$

Some permutations could be sorted using only one stack. These are exactly the permutations avoiding 231.



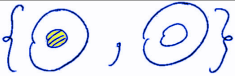
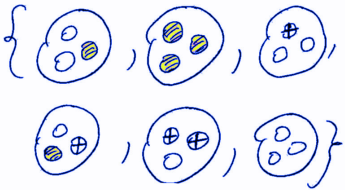


Enumerated by Catalan Numbers: $1, 1, 2, 5, 14, 42, \ldots$
Knuth sorts such permutations in linear time using only one stack.

Patterns in other structures

1. Patterns in words [Axel Thue, 1906, ...]
2. Patterns in set partitions [Martin Klazar, 1996]
3. Patterns in inversion sequences [Corteel, Martinez, Savage, Weselcouch, 2016] [Mansour, Shattuck, 2015]
4. Patterns in graphs, X-free graphs
5. Patterns in DNA, in complex networks, ...

Meaning of “pattern” depends on the context!

$\mathcal{A}_i$	$\lvert\mathcal{A}_i\rvert$	$\lvert\mathcal{A}_i($-  $)\rvert$
$\mathcal{A}_1 =$ 	2	1
$\mathcal{A}_2 =$ 	3	1
$\mathcal{A}_3 =$ 	6	3

Popularity of  1, 2, 5, ...

Background and motivations

Our motivations

- New objects/patterns
- New interesting properties (enumerated by existing/new sequences, and bijectively linked to existing structures)

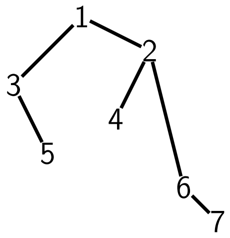
My personal motivations

- I'm looking for interesting connections between enumerative/bijective combinatorics and structure/dynamics of complex networks (internet, brain, proteins, social), certain graph coloration problems.

Treeshelves

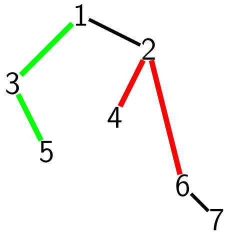
are ordered binary increasing trees where every child is connected to its parent by a left or a right link.

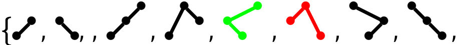
Treeshelves, patterns and permutations



Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.

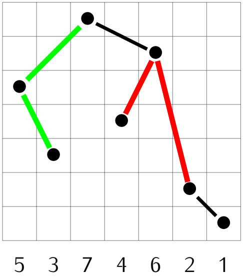
Treeshelves, patterns and permutations

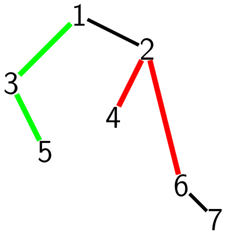


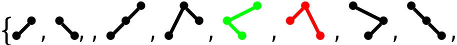
Patterns {  }.

Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.

Treeshelves, patterns and permutations





Patterns {  }.

Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.

Françon bijection

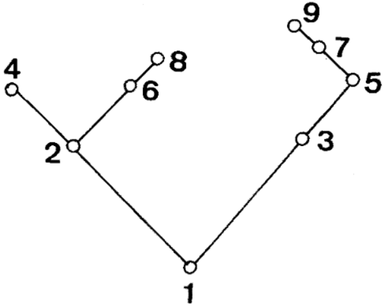


Figure 2.
Arbre binaire décroissant correspondant au mot 4 2 6 8 1 3 9 7 5.

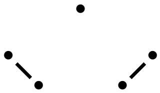
Bijection: Treeshelves ↔ Permutations

[Jean Françon, 1976]

Analytic enumeration of treeshelves

Labeled objects

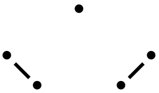
Unlabeled objects



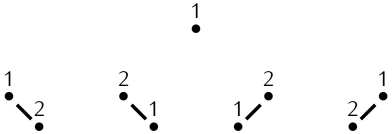
Generating function $\sum a_n x^n$

Labeled objects

Unlabeled objects



Labeled



Generating function $\sum a_n x^n$

Generating function $\sum b_n \frac{x^n}{n!}$

a_n and b_n count objects of size n.

$b_n = a_n n!$

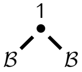
Boxed product of labeled objects

Objects		Generating function
$\mathcal{A}$		A
$\mathcal{B}$		B
$\mathcal{B} \star \mathcal{A}$	pairs with relabeling	$A \cdot B$
$\mathcal{A}^{\square} \star \mathcal{B}$	pairs with relabeling, the smallest label goes to $\mathcal{A}$	$\int_0^z \partial_t A(t) B(t)\, \mathrm{d}t$

[Flajolet and Sedgewick, Analytic combinatorics Theorem II.5]

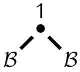
Enumeration of treeshelves

$\mathcal{Z} =$ 

$\mathcal{B} =$ EMPTY or 

Enumeration of treeshelves

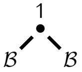
$\mathcal{Z} = \overset{1}{\bullet}$

$\mathcal{B} =$ EMPTY or 

$$\mathcal{B} = \epsilon + \mathcal{Z}^{\square} \star \left(\mathcal{B} \star \mathcal{B}\right)$$

Enumeration of treeshelves

$\mathcal{Z} = \overset{1}{\bullet}$

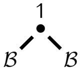
$\mathcal{B} =$ EMPTY or 

$$\mathcal{B} = \epsilon + \mathcal{Z}^{\Box} \star \left(\mathcal{B} \star \mathcal{B}\right)$$

$$B(z) = 1 + \int_0^z B^2(t)\,\mathrm{d}t,\; B(0) = 1$$

Enumeration of treeshelves

$\mathcal{Z} = \overset{1}{\bullet}$

$\mathcal{B} =$ EMPTY or 

$\mathcal{B} = \epsilon + \mathcal{Z}^{\square} \star \left(\mathcal{B} \star \mathcal{B}\right)$

$B(z) = 1 + \int_0^z B^2(t)\,\mathrm{d}t,\ B(0) = 1$

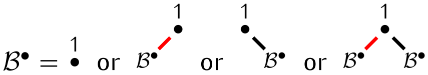
$B(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} n! \frac{z^n}{n!}$

i.e. the exponential generating function for $n!$

Left children in treeshelves

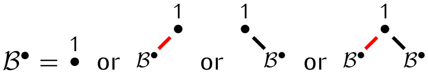
Left children in treeshelves

(non empty treeshelves)



Left children in treeshelves

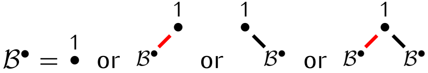
(non empty treeshelves)



$$\mathcal{B}^\bullet = \mathcal{Z} + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star (\mathcal{B}^\bullet)^2$$

Left children in treeshelves

(non empty treeshelves)



$$\mathcal{B}^\bullet = \mathcal{Z} + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star (\mathcal{B}^\bullet)^2$$

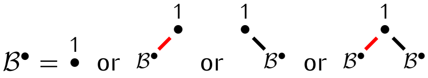
Use y for left children.

$$B^\bullet(z,y) = z + {\color{red}y} \int_0^z B^\bullet(t,y)\,\mathrm{d}t + \int_0^z B^\bullet(t,y)\,\mathrm{d}t + {\color{red}y} \int_0^z (B^\bullet(t,y))^2\,\mathrm{d}t$$

Initial condition $B^\bullet(0,y) = 0$

Left children in treeshelves

(non empty treeshelves)



$$\mathcal{B}^\bullet = \mathcal{Z} + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star \mathcal{B}^\bullet + \mathcal{Z}^\square \star (\mathcal{B}^\bullet)^2$$

Use y for left children.

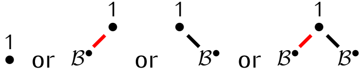
$$B^\bullet(z,y) = z + {\color{red}y} \int_0^z B^\bullet(t,y)\,\mathrm{d}t + \int_0^z B^\bullet(t,y)\,\mathrm{d}t + {\color{red}y} \int_0^z (B^\bullet(t,y))^2\,\mathrm{d}t$$

Initial condition $B^\bullet(0,y) = 0$

$$B^\bullet(z,y) = \frac{1-\mathrm{e}^{z(y-\mathbf{1})}}{\mathrm{e}^{z(y-\mathbf{1})}-y}$$

Left children in treeshelves

(non empty treeshelves)

$\mathcal{B}^\bullet =$ 

$$\mathcal{B}^\bullet = \mathcal{Z} + \mathcal{Z}^\Box \star \mathcal{B}^\bullet + \mathcal{Z}^\Box \star \mathcal{B}^\bullet + \mathcal{Z}^\Box \star (\mathcal{B}^\bullet)^2$$

Use y for left children.

$$B^\bullet(z,y) = z + y \int_0^z B^\bullet(t,y)\,\mathrm{d}t + \int_0^z B^\bullet(t,y)\,\mathrm{d}t + y \int_0^z (B^\bullet(t,y))^2\,\mathrm{d}t$$

Initial condition $B^\bullet(0,y) = 0$

$$B^\bullet(z,y) = \frac{1-\mathrm{e}^{z(y-\mathbf{1})}}{\mathrm{e}^{z(y-\mathbf{1})}-y}$$

$$B(z,y) = 1 + B^\bullet(z,y) = \frac{1-y}{\mathrm{e}^{z(y-\mathbf{1})}-y}$$

$$B(z,y) = 1 + B^{\bullet}(z,y) = \frac{1-y}{e^{z(y-1)} - y}$$

- Left children distribution in treeshelves has exponential generating function $B(z,y)$
  - shift of Eulerian numbers A008292
- Left children popularity corresponds to $\partial_y B(z,y)|_{y=1} = \frac{z^2}{2z^2-4z+2}$
  - Lah numbers A001286.

well known results
see [Petersen, Eulerian numbers, 2015]

Treeshelves avoid patterns

T-patterns, patterns in Threshelfs

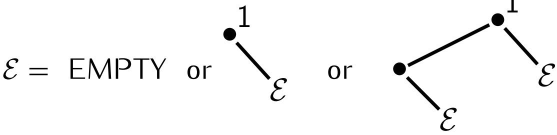
$\mathcal{Z} =$ 

$\mathcal{E}$ denotes treeshelves avoiding 

T-patterns, patterns in Threshelfs

$\mathcal{Z} = \overset{1}{\bullet}$

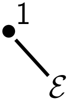
$\mathcal{E}$ denotes treeshelves avoiding 

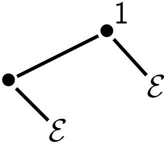


T-patterns, patterns in Threshelfs

$\mathcal{Z} =$ 

$\mathcal{E}$ denotes treeshelves avoiding 

$\mathcal{E} =$ EMPTY or 

or 

$$\mathcal{E} = \epsilon + \mathcal{Z}^{\square} \star \mathcal{E} + \left(\mathcal{Z}^{\square} \star \mathcal{E}\right)^{\square} \star \left(\mathcal{Z}^{\square} \star \mathcal{E}\right)$$

Boxed product → integral equation
The equation + initial conditions → generating function.

Treeshelves avoiding a size 3 pattern

$\mathcal{B}(P)$ denotes treeshelves avoiding a t-pattern

Pattern P	Sequence counting $\mathcal{B}(P)$	OEIS
	$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$	A000110 (Bell)
	$1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots$	A000111 (Euler)
	$1, 1, 2, 5, 16, 64, 308, 1730, 11104, 80176, \ldots$	A131178

Left children distribution in treeshelves avoiding a size 3 pattern

$\mathcal{B}(P)$ denotes treeshelves avoiding a t-pattern

y corresponds to left childrens

Pattern P	Generating function for $\mathcal{B}(P)$
	$e^{\frac{e^{zy}-1}{y}}$
	$\frac{2y-1}{y\cosh\left(z\sqrt{-2y+1}+\ln\left(\frac{1}{y}\left(y+\sqrt{-2y+1}-1\right)\right)\right)+y}$
	$\frac{-2}{1+y-\sqrt{y^2+1}\coth\left(\frac{z\sqrt{y^2+1}}{2}\right)}$

Left children popularity I

sequences

Pattern P	Popularity of left children in $\mathcal{B}(P)$	
	$1, 5, 23, 109, 544, 2876, 16113, \ldots$	*A278677*
	$1, 4, 19, 94, 519, 3144, 20903, 151418, \ldots$	*A278678*
	$1, 5, 24, 128, 770, 5190, 38864, 320704, \ldots$	*A278679*

Left children popularity II

egfs, asymptotics, probability

Pattern P	Generating functions	Asymptotics
	$(ze^z - e^z + 1)\, e^{e^z-1}$	$\sqrt{n}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)}-n-1}$
	$\frac{-\sin z+1+(z-1)\cos z}{(1-\sin z)^2}$	$\frac{8(\pi-2)}{\pi^3}\, n^2 \left(\frac{2}{\pi}\right)^n$
	$\frac{e^{\sqrt{2}z}(4z-4)-\left(\sqrt{2}-2\right)e^{2\sqrt{2}z}+\sqrt{2}+2}{\left(\left(\sqrt{2}-2\right)e^{\sqrt{2}z}+2+\sqrt{2}\right)^2}$	$n\left(\frac{\sqrt{2}}{\ln\left(2\sqrt{2}+3\right)}\right)^{n+1}$

W is the Lambert function, i.e. $W(n)$ is the unique solution of $W(n)\cdot e^{W(n)} = n$

Asymptotics could be used to estimate the probability that a randomly selected link is left.

Popularity of left children III

interesting facts

Moreover,

Left children popularity in $\mathcal{B}$() of size n equals

$$(n+1)b_n - b_{n+1}$$

where b_n is the *n-th* Bell number.

Popularity of left children III

interesting facts

Moreover,

Left children popularity in $\mathcal{B}\Bigg($  $\Bigg)$ of size n equals

$$(n+1)b_n - b_{n+1}$$

where b_n is the *n-th* Bell number.

Left children popularity in $\mathcal{B}\Bigg($  $\Bigg)$ of size n equals

$$(n+1)e_n - e_{n+1}$$

where e_n is the shifted Euler number defined by the e.g.f. $\frac{1}{1-\sin(z)}$.

Is it easy to calculate coefficients ?

P-recursivity

D-finite functions

are solutions of ordinary differential equations with polynomial coefficients

P-recursive sequence

Sequence a_n is P-recursive if $\exists k$ and polynomials $p_0, p_1, \ldots, p_k$ such that

$$p_0(n) \cdot a_n = p_1(n) \cdot a_{n-1} + p_2(n) \cdot a_{n-2} + \ldots + p_k(n) \cdot a_{n-k}$$

- D-finite function generates p-recursive sequence and vice versa. $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Fast coeff. calculations. Nice properties. Important notions.

For more info see, for example, Cyril Banderier's talk
`https://www.irif.fr/~poulalho/ALEA09/slides/banderier.pdf`

Popularity sequences are not P-recursive

Pattern P	Generating functions	Asymptotics
	$(ze^z - e^z + 1)\, e^{e^z-1}$	$\sqrt{n}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)}-n-1}$
	$\frac{-\sin z+1+(z-1)\cos z}{(1-\sin z)^2}$	$\frac{8(\pi-2)}{\pi^3} n^2 \left(\frac{2}{\pi}\right)^n$
	$\frac{e^{\sqrt{2}z}(4z-4)-\left(\sqrt{2}-2\right)e^{2\sqrt{2}z}+\sqrt{2}+2}{\left(\left(\sqrt{2}-2\right)e^{\sqrt{2}z}+2+\sqrt{2}\right)^2}$	$n\left(\frac{\sqrt{2}}{\ln\left(2\sqrt{2}+3\right)}\right)^{n+1}$

The functions above are not D-finite:

- e^{e^z-1} is not D-finite because it grows too fast
- D-finite $\Rightarrow$ finitely many singularities

[Flajolet, Gerhold, Salvy, 2005]

Bijections

Bijections

Pattern P	Treeshelfs avoiding P are in bijection with...
	Set partitions
	Unordered (non-plane) binary increasing trees
	Unordered binary increasing trees where the nodes of outdegree 1 come in 2 colors A131178

Unordered ↔ Ordered

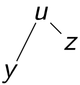
Theorem

There is a bijection between unordered binary increasing trees with $n+1$ nodes and the set $\mathcal{B}_n$( *) of t-shelves of size n avoiding the pattern*  *.*

Standard representation

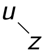
Standard representation of an unordered (non-plane) tree

- Nodes with two children:



, when $z < y$

- Nodes with only one child:



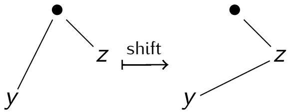
.

Standard representation is a treeshelf

Shift

Shift a node y of treeshelf under two conditions:

- y is a left child and it has a right sibling, say z; and
- z in turn does not have a left child and its label is smaller than that of y.



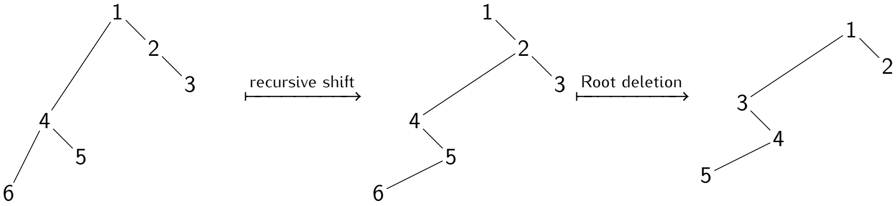
Shift of a treeshelf is defined recursively by shifting, in order, the right subtree, the root, and then the left subtree.

Unordered ↔ Ordered

Theorem

There is a bijection between unordered binary increasing trees with $n+1$ nodes and the set $\mathcal{B}_n\big($$\big)$.

Proof illustration

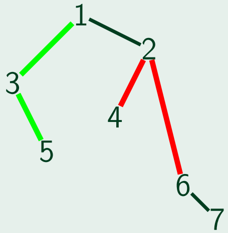


Conclusion

Our results:

- Treeshelfs avoiding pattern of size 3
  - Known sequences
  - Analytic enumeration
  - Bijections
- Distribution of left children in treeshelfs avoiding patterns of size 3
  - Bivariate generating functions with respect to the number left children and size.
- Popularity of left children in treeshelfs avoiding patterns of size 3
  - New sequences!
  - Not P-recursive, MC-finite (!?)
  - Asymptotics provided

Patterns in treeshelves



Paper: `https://arxiv.org/abs/1611.07793v1`

Slides: `http://kirgizov.link/talks/dijon-2017.pdf`