Qubonacci words and related structures

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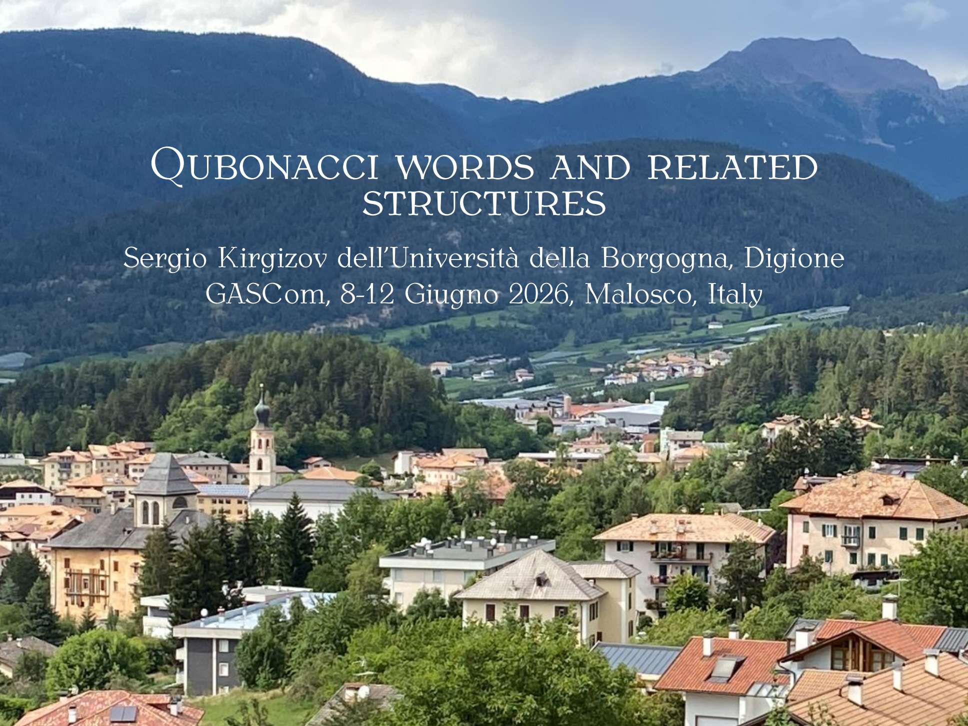

1. Poetry metrics, Knuth-Fibonacci words

2. Fibonacci, Tribonacci, $6/7$-bonacci, $\sqrt{2}$-bonacci ?

3. Structure and interpretation of q-decreasing words:
   natural, rational and real cases

4. Pecular growth function

5. Gray codes

6. Lattice and other structures

7. Open questions

Poetry metrics,
Knuth-Fibonacci words

Poetry metrics

da DUM   da DUM   da DUM ...

DUM da   DUM da   DUM da ...

da da DUM   da da DUM ...

da DUM da   da DUM da   da DUM da ...

DUM da da   DUM da da ...

Poetry metrics

In Ancient Greek, Latin, Sanskrit and some other languages there are two types of syllables: short and long.

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⏑	⏑ ⏑	⏑ ⏑ ⏑	⏑ ⏑ ⏑ ⏑	⏑ ⏑ ⏑ ⏑ ⏑
	—	⏑ —	⏑ ⏑ —	⏑ ⏑ ⏑ —
		— ⏑	⏑ — ⏑	⏑ ⏑ — ⏑
			— ⏑ ⏑	⏑ — ⏑ ⏑
			— —	⏑ — —
				— ⏑ ⏑ ⏑
				— ⏑ —
				— — ⏑

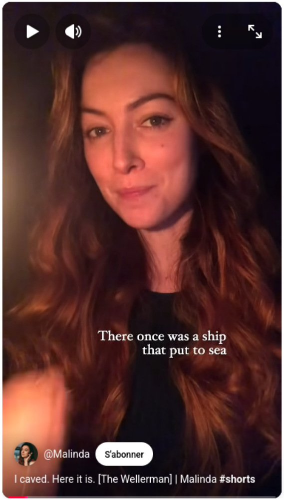
In accentual-syllabic verses, in English and Russian poetry for instance, we have stressed (accentuated) and nonstressed syllables. Compare with long and short.

The Wellerman

There ***once*** *was a* ***ship*** *that put* ***to*** *sea*

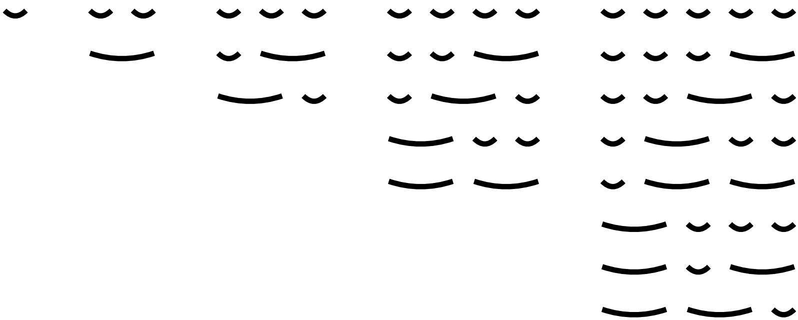
The ***name*** *of the* ***ship*** *was the* ***Billy*** *O'* ***Tea..***

New Zealand Folk Song



Poetry metrics

A metre is a rhythmic structure of a verse, a pattern of alternating short and long (stressed/nonstressed) syllables.



Poetry metrics

A metre is a rhythmic structure of a verse, a pattern of alternating short and long (stressed/nonstressed) syllables.

0

0 0
1 0

0 0 0
0 1 0
1 0 0

0 0 0 0
0 0 1 0
0 1 0 0
1 0 0 0
1 0 1 0

0 0 0 0 0
0 0 0 1 0
0 0 1 0 0
0 1 0 0 0
0 1 0 1 0
1 0 0 0 0
1 0 0 1 0
1 0 1 0 0

Encode short as 0, long as 10.
Obtain words avoiding 11,
ending with 0.
How many metres we have ?

Poetry metrics

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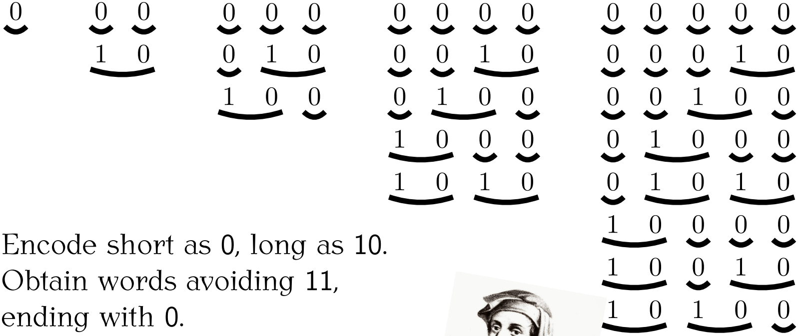

Encode short as 0, long as 10.
Obtain words avoiding 11,
ending with 0.
How many metres we have ?

1, 2, 3, 5, 8,...

Knuth-Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers.

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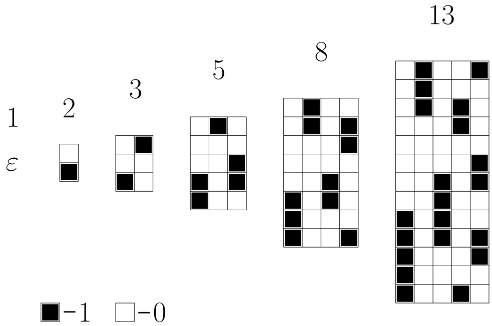
Words avoiding 11 are counted by Fibonacci

$$a_n = a_{n-1} + a_{n-2}$$



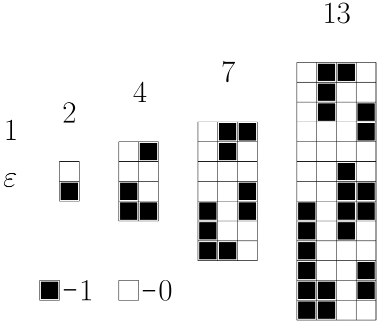
Words are listed in Gray order (consecutive differs in only 1 bit)

Knuth-Fibonacci words

Words avoiding 1^k are counted by generalized k-step Fibonacci numbers.

Words avoiding 111 are counted by Tribonacci

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$



Words are listed in Gray order (consecutive differs in only 1 bit)

Knuth-Fibonacci words literature

- The Art of Computer Programming, Vol. 3: Sorting, and Searching, 2 ed. (page 286), 1998, Donald Knuth
- Matters Computational (Section 14.2), 2010, Jörg Arndt https://www.jjj.de/fxt/fxtbook.pdf
- Generalized Fibonacci cubes are mostly Hamiltonian Jenshiuh Liu, Wen-Jing Hsu, Moon Jung Chung, 1994
- Gray codes for A-free strings. Matthew B. Squire, 1996
- A loopless generation of bitstrings without p consecutive ones Vincent Vajnovszki, 2001
- An O(1) time algorithm for generating Fibonacci strings Kenji Mikawa and Ishiro Semba, 2005

Fibonacci, Tribonacci,

Fibonacci, Tribonacci, Tetranacci, Pentanacci…

Fibonacci, Tribonacci, Tetranacci, Pentanacci…

7/3-bonacci ?

Fibonacci, Tribonacci, Tetranacci, Pentanacci…

7/3-bonacci ?

$\sqrt{2}$-bonacci ?

ϕ-bonacci ?

q-decreasing words

An n-length binary word is *q-decreasing*, $q \in \mathbb{R}^+$, if every of its length maximal factors of the form $0^a 1^b$ satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1 \left| \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} \right| 0 \cdots$$

Let $\mathcal{W}_{q,n}$ be the set of such words of length n, $\mathcal{W}_q = \bigcup_{n \in \mathbb{N}} \mathcal{W}_{q,n}$.

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Ex.

11100101**0110**001 is not 2-decreasing $(2 \cdot 1 \not> 2)$

01111 is not π-decreasing $(\pi \cdot 1 \not> 4)$

001111 is π-decreasing $(\pi \cdot 2 > 4)$

Natural q

1-decreasing words, $\mathcal{W}_1$

In a 1-decreasing word every run of 0s is immediately followed by a strictly shorter run of 1s.

$$\cdots 1 \,\Big|\, \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \,\Big|\, 0\cdots \qquad a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

0	00	000	0000	…
1	10	001	0001	
	11	100	0010	
		110	1000	
		111	1001	
			1100	
			1110	
			1111	

A typical word in $\mathcal{W}_1$ looks like:

$$1 \ldots 1\, \sigma_1\, \sigma_2 \ldots \sigma_\ell$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0

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Admissible suffixes
0
0 **<u>01</u>**

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Admissible suffixes
0
0 **<u>01</u>**
00 **<u>01</u>** 1

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Admissible suffixes
0
0 **<u>01</u>**
00 **<u>01</u>** 1
000 **<u>01</u>** 11

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Admissible suffixes
0
0 <u>**01**</u>
00 <u>**01**</u> 1
000 <u>**01**</u> 11
0000 <u>**01**</u> 111

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00000 **01** 1111

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Admissible suffixes
0
0 **01**
00 **01** 1
000 **01** 11
0000 **01** 111
00000 **01** 1111
...
$\overbrace{0\cdots00}^{1+i\text{ zeros}}\,\underbrace{1\cdots11}_{i\text{ ones}}$

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Admissible suffixes	Generating functions
0	$\frac{1}{1-x}$ - sequence of 1s.
0 **01**	
00 **01** 1	
000 **01** 11	
0000 **01** 111	
00000 **01** 1111	
...	
$\overbrace{0\cdots00}^{1+i \text{ zeros}}\underbrace{1\cdots11}_{i \text{ ones}}$	

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Admissible suffixes	Generating functions
0	$\dfrac{1}{1-x}$ - sequence of 1s.
$0\,\underline{\mathbf{01}}$	$\dfrac{x}{1-x^2}$ - g.f. for admissible suffixes.
$00\,\underline{\mathbf{01}}\,1$	
$000\,\underline{\mathbf{01}}\,11$	
$0000\,\underline{\mathbf{01}}\,111$	
$00000\,\underline{\mathbf{01}}\,1111$	
$\ldots$	
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$000\,\underline{\mathbf{01}}\,11$	
$0000\,\underline{\mathbf{01}}\,111$	$\frac{1}{1-x} \cdot \frac{1}{1-\frac{x}{1-x^2}}$ - sequence of 1s followed
$00000\,\underline{\mathbf{01}}\,1111$	by a sequence of admissible suffixes.
$\ldots$	

$$\overbrace{0\cdots00}^{1+i \text{ zeros}}\underbrace{1\cdots11}_{i \text{ ones}}$$

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$\ldots$	by a sequence of admissible suffixes.

$$\overbrace{0 \cdots 00}^{1+i \text{ zeros}}\,\underbrace{1 \cdots 11}_{i \text{ ones}}$$

$$W_1(x) = \frac{1+x}{1-x-x^2}$$

Bijection with words avoiding 11

Words avoiding 11	1-decreasing words
ε	ε
1	1
0 [word of size n - 1]	[word of size n - 1] 0
1 0 [word of size n-2]	[....... 0] 0 1 [1 ... 1] concatenation of these gives a word of size n - 2, ending with 011..1 (with 0 followed by a possibly empty sequence of 1s) 1 1 [1 1 .... 1] word of size n - 2, only 1s

Bijection with words avoiding 11

First define a map $\psi : \{0,1\}^n \to \{0,1\}^{n+2}$.

$$\psi(w) = \begin{cases} v0\,01\,1^k & \text{if } w = v01^k, k \geq 0 \\ 11\,1^n & \text{otherwise } w = 1^n \end{cases}$$

Now, construct a length-preserving bijection ϕ that maps binary words avoiding 11 to 1-decreasing words.

$$\phi(w) = \begin{cases} \varepsilon & \text{if } w = \varepsilon, \\ 1 & \text{if } w = 1, \\ \psi\big(\phi(v)\big) & \text{if } w = 10v, \\ \phi(v)0 & \text{if } w = 0v. \end{cases}$$

2-decreasing words, $\mathcal{W}_2$

$$\cdots 1 \,\Big|\, \underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b} \,\Big|\, 0\cdots \qquad \text{where } 2a > b \text{ or } a = 0$$

Let's count!

n	1	2	3	4	$\cdots$
	2	4	7	13	Tribonacci

0	00	000	0000
1	01	001	0001
	10	010	0010
	11	100	0011
		101	0100
		110	0101
		111	1000
			1001
			1010
			1100
			1101
			1110
			1111

...

Words avoiding 1^{q+1}	q-decreasing words
ε	ε
1	1
11	11
• • •	• • •
1^q	1^q
0 [word of size n - 1]	[word of size n - 1] 0
1 0 [word of size n-2]	[word of size n-2] 0 1
1 1 0 [word of size n-3]	[word of size n-3] 0 1 1
• • •	• • •
1^{q-1} 0 [word of size n-q]	[word of size n-q] 0 1^{q-1}
1^q 0 [word of size n-(q+1)]	[..... 0] 0 1^q [1 ... 1]

concatenation of these gives
a word of size $n-(q+1)$, ending with 011..1
(with 0 followed by a possibly empty sequence of 1s)

1^{q+1} [1 1 .... 1]

word of size $n-(q+1)$, only 1s

Bijection with words avoiding 1^{q+1}

First define a map $\psi : \{0,1\}^n \to \{0,1\}^{n+q+1}$.

$$\psi(w) = \begin{cases} v001^{k+q} & \text{if } w = v01^k, k \geq 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Now, construct a length-preserving bijection ϕ that maps binary words avoiding 1^{q+1} to q-decreasing words.

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0,q], \\ \psi\big(\phi(v)\big) & \text{if } w = 1^q0v, \\ \phi(v)01^k & \text{if } w = 1^k0v \text{ and } k \in [0,q-1]. \end{cases}$$

$av.111$	$\xrightarrow{\phi}$	2-dec.
1100		0011
1101		1111
1001		1001
1000		0001
1010		0101
1011		1101
0011		1100
0010		0100
0000		0000
0001		1000
0101		1010
0100		0010
0110		1110

The bijection does not preserve *Graycodeness*.

q-decreasing words with natural q

- Bijections between q-decreasing words and words avoiding factors 1^{q+1}.
- Efficient generation and Gray codes
- Solved Eğecioğlu-Iršič conjecture (Hamiltonian path always exists in Fibonacci-run graphs)

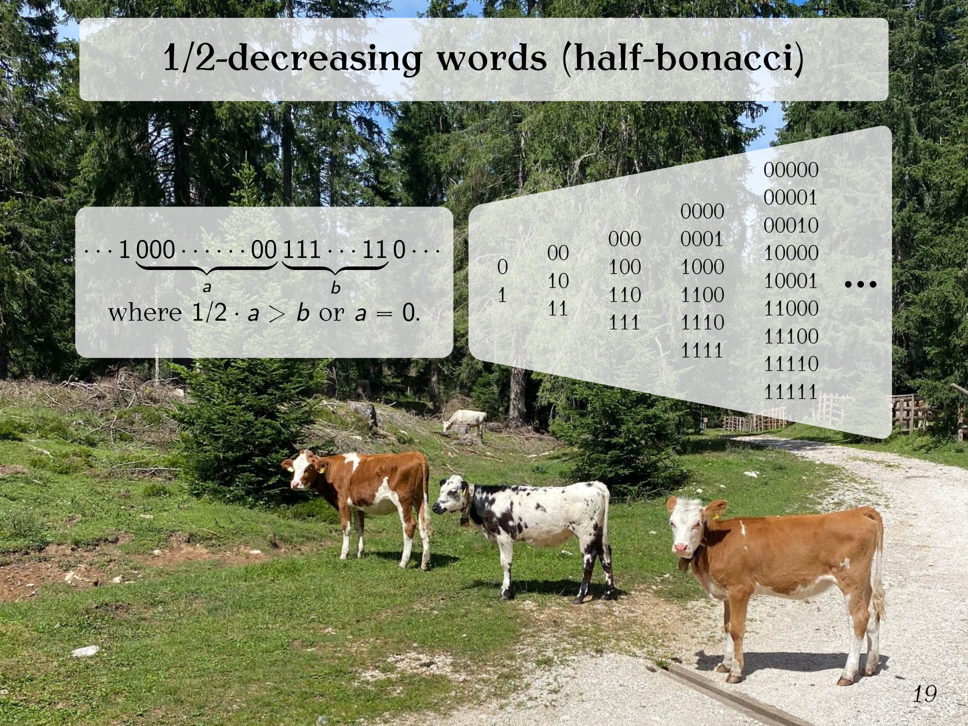
Gray codes for Fibonacci q-decreasing words.
Jean-Luc Baril, Sk and Vincent Vajnovszki
Theoretical Computer Science, 2022, https://arxiv.org/abs/2010.09505

Fibonacci-run graphs I: Basic properties. Ömer Eğecioğlu and Vesna Iršič
Discrete Applied Mathematics, 2021, https://arxiv.org/abs/2010.05518

Asymptotic bit frequency in Fibonacci words. BKV, GASCom 2022
https://kirgizov.link/talks/gascom2022.pdf
Pure Mathematics and Applications, 2022, https://arxiv.org/abs/2106.13550

Qubonacci words. BKV
Permutations patterns 2021, https://kirgizov.link/talks/qubonacci.pdf

Rational q



1/2-decreasing words (half-bonacci)

$$\cdots 1\underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b}0\cdots$$

where $1/2 \cdot a > b$ or $a = 0$.

0	00	000	0000	00000	...
1	10	100	0001	00001	
	11	110	1000	00010	
		111	1100	10000	
			1110	10001	
			1111	11000	
				11100	
				11110	
				11111	

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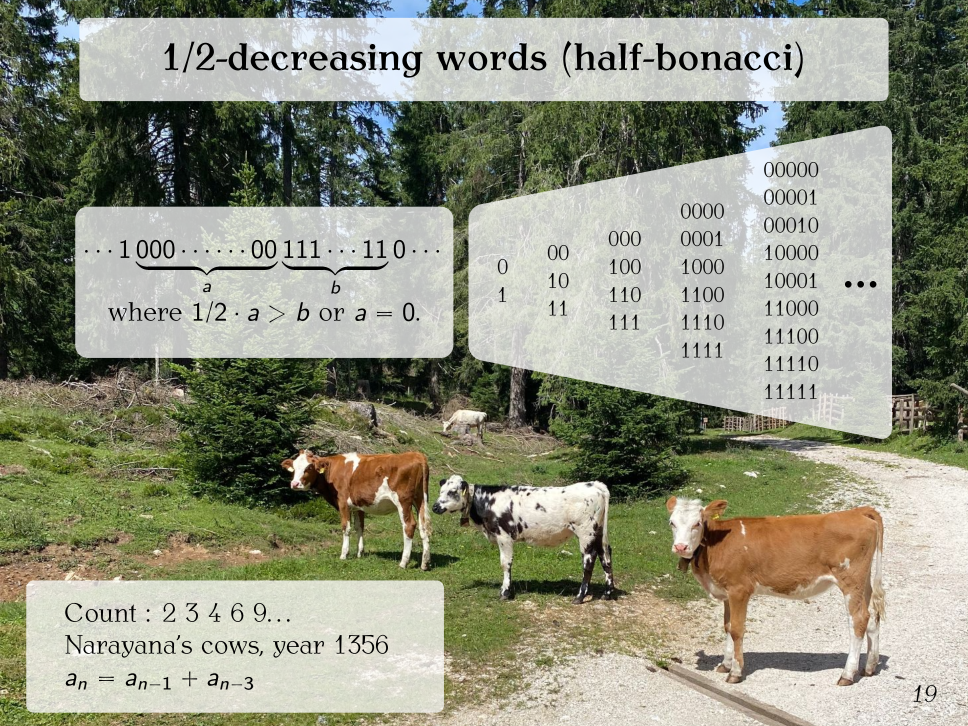
$$\cdots 1\underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b}0\cdots$$

where $1/2 \cdot a > b$ or $a = 0$.

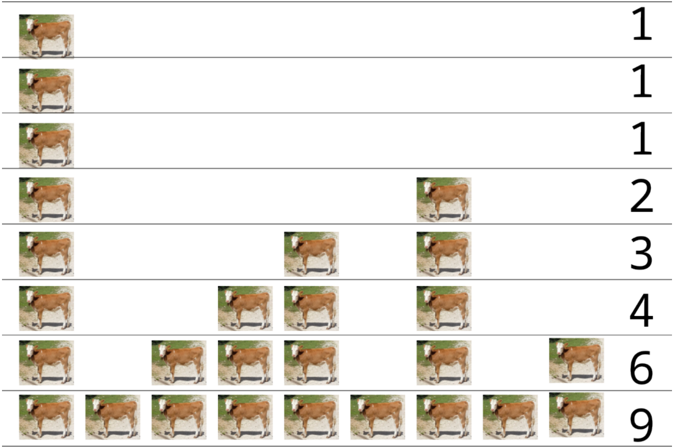
0	00	000	0000	00000	…
1	10	100	0001	00001	
	11	110	1000	00010	
		111	1100	10000	
			1110	10001	
			1111	11000	
				11100	
				11110	
				11111	

Count : 2 3 4 6 9…
Narayana's cows, year 1356
$a_n = a_{n-1} + a_{n-3}$



Narayana's cows

Start with one cow, at age 0.
Each cow gives birth to another cow every year starting from the age of three.



Construction

Example $q = 1/2$

Every word from $\mathcal{W}_{1/2}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0

Example $q = 1/2$

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Admissible suffixes
0
0 $\underline{\mathbf{001}}$

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Admissible suffixes
0
0 $\underline{\mathbf{001}}$
000 $\underline{\mathbf{001}}$ 1

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Admissible suffixes
0
$0\,\underline{\mathbf{001}}$
$000\,\underline{\mathbf{001}}\,1$
$00000\,\underline{\mathbf{001}}\,11$

Example $q = 1/2$

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0
$0\,\underline{\mathbf{001}}$
$000\,\underline{\mathbf{001}}\,1$
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$0\,\underline{\mathbf{001}}$
$000\,\underline{\mathbf{001}}\,1$
$00000\,\underline{\mathbf{001}}\,11$
$0000000\,\underline{\mathbf{001}}\,111$
$000000000\,\underline{\mathbf{001}}\,1111$

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Every word from $\mathcal{W}_{1/2}$ looks like

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where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
0 **<u>001</u>**
000 **<u>001</u>** 1
00000 **<u>001</u>** 11
0000000 **<u>001</u>** 111
000000000 **<u>001</u>** 1111
$\cdots$
$\overbrace{0 \cdots 00}^{1+2i \text{ zeros}} \; \underbrace{1 \cdots 11}_{i \text{ ones}}$

Example $q = 1/2$

Every word from $\mathcal{W}_{1/2}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
$0\,\underline{\mathbf{001}}$
$000\,\underline{\mathbf{001}}\,1$
$00000\,\underline{\mathbf{001}}\,11$
$0000000\,\underline{\mathbf{001}}\,111$
$000000000\,\underline{\mathbf{001}}\,1111$
$\cdots$

$$\overbrace{0\cdots00}^{1+2i\text{ zeros}}\ \underbrace{1\cdots11}_{i\text{ ones}}$$

Model polynomial $P_{1/2}(y,z) = z$ encodes the initial admissible suffix 0.

Example $q = 1/2$

Every word from $\mathcal{W}_{1/2}$ looks like

$$1 \cdots 1 \, \sigma_1 \, \sigma_2 \, \cdots \, \sigma_\ell ,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
0 **<u>001</u>**
000 **<u>001</u>** 1
00000 **<u>001</u>** 11
0000000 **<u>001</u>** 111
000000000 **<u>001</u>** 1111
...

$$\overbrace{0 \cdots 00}^{1+2i \text{ zeros}} \; \underbrace{1 \cdots 11}_{i \text{ ones}}$$

Model polynomial $P_{1/2}(y, z) = z$ encodes the initial admissible suffix 0.

Spawning infix 001 is encoded by $z^2 y$.

Admissible suffixes are constructed iteratively by injecting the spawning infix 001 just after the last 0 in already constructed suffixes.

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001
0 **<u>00011</u>**

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001
0 <u>**00011**</u>
00 <u>**00011**</u> 1

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Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

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Admissible suffixes
0
001
0 <u>**00011**</u>
00 <u>**00011**</u> 1
0000 <u>**00011**</u> 11

Example $q = 2/3$

Every word from $\mathcal{W}_{2/3}$ looks like

$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell\,,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001
0 **<u>00011</u>**
00 **<u>00011</u>** 1
0000 **<u>00011</u>** 11
00000 **<u>00011</u>** 111

Example $q = 2/3$

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$$1 \cdots 1\, \sigma_1\, \sigma_2\, \cdots\, \sigma_\ell,$$

where σ_i is an element from the set of admissible suffixes.

Admissible suffixes
0
001
0 $\underline{\mathbf{00011}}$
00 $\underline{\mathbf{00011}}$ 1
0000 $\underline{\mathbf{00011}}$ 11
00000 $\underline{\mathbf{00011}}$ 111

$$\overbrace{0 \cdots 00}^{1+\left\lfloor \frac{i}{q} \right\rfloor \text{ zeros}} \quad \underbrace{1 \cdots 11}_{i \text{ ones}}$$

Model polynomial $P_{2/3}(y, z) = z + z^2 y$ encodes initial admissible suffixes 0 and 001.

Spawning infix 00011 is encoded by $z^3 y^2$.

Admissible suffixes are constructed iteratively by injecting the spawning infix 00011 just after the last 0 in already constructed suffixes.

q	Model polynomial	Spawning infix g.f.
$1/k$	z	$z^k y$
2	$z + zy$	zy^2
$2/3$	$z + z^2 y$	$z^3 y^2$
$3/2$	$z + zy + z^2 y^2$	$z^2 y^3$
$3/4$	$z + z^2 y + z^3 y^2$	$z^4 y^3$
$3/5$	$z + z^2 y + z^4 y^2$	$z^5 y^3$
...	...	...

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$.

Spawning infix, $\underbrace{0 \cdots 00}_{d}\underbrace{11 \cdots 1}_{c}$, has g.f. $z^d y^c$.

Model polynomial is $P_{q=\frac{c}{d}}(y, z) = \sum_{i=0}^{c-1} z^{1+\left\lfloor \frac{i}{q} \right\rfloor} y^i$.

Generating function

Theorem (K. 2022)

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$. The generating function

$$W_q(y,z) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty} w_{r,i} z^r y^i$$

where $w_{r,i}$ is number of words from $\mathcal{W}_q$ of length $r+i$ containing exactly r zeros and i ones is

$$W_q(y,z) = \frac{1 - z^d y^c}{(1-y)\big(1 - z^d y^c - P_q(y,z)\big)},$$

where $P_q(y,z)$ is the model polynomial of q.

Linear recurrence with 0-1 coefficients

Theorem (K. 2022)

Let $q \in \mathbb{Q}^+$ be represented by the irreducible fraction $\frac{c}{d}$. The number of n-length binary words from $\mathcal{W}_{q,n}$, denoted by w_n, can be expressed as

$$w_n = \sum_{j \in J} w_{n-j} + w_{n-(c+d)},$$

where J is the set of powers from $P_q(x,x)$.

For example, when $q = \frac{3}{2}$,
we have $P_{\frac{3}{2}}(x,x) = x + x^2 + x^4$, and $J = \{1,2,4\}$.

With appropriate initial conditions:)

q	Sequence	Recurrence relation	OEIS (with shifts)
1/5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, ...	$w_n = w_{n-1} + w_{n-6}$	Compositions (ordered partitions) of n into 1s and 6s. A5708
1/4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, ...	$w_n = w_{n-1} + w_{n-5}$	C. into 1s and 5s. A3520
1/3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, ...	$w_n = w_{n-1} + w_{n-4}$	C. into 1s and 4s. A3269
2/5	1, 2, 3, 4, 6, 9, 13, 18, 26, 38, 55, 79, ...	$w_n = w_{n-1} + w_{n-4} + w_{n-7}$	C. into 1s, 4s and 7s. Not in OEIS.
1/2	1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, ...	$w_n = w_{n-1} + w_{n-3}$	Narayana's cows, A930
3/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-6} + w_{n-8}$	NEW
2/3	1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5}$	C. into 1s, 3s and 5s, A60961
3/4	1, 2, 3, 5, 8, 13, 21, 33, 53, 85, 136, 218, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7}$	C. into 1s, 3s, 5s and 7s, A117760
4/5	1, 2, 3, 5, 8, 12, 19, 30, 46, 72, 113, 176, ...	$w_n = w_{n-1} + w_{n-3} + w_{n-5} + w_{n-7} + w_{n-9}$	NEW
1	1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...	$w_n = w_{n-1} + w_{n-2}$	Fibonacci numbers, A45
5/4	1, 2, 4, 7, 13, 23, 42, 75, 136, 244, 441, 794, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-8} + w_{n-9}$	NEW
4/3	1, 2, 4, 7, 13, 23, 42, 75, 136, 245, 443, 799, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-6} + w_{n-7}$	NEW
3/2	1, 2, 4, 7, 13, 23, 42, 76, 138, 250, 453, 821, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5}$	NEW
5/3	1, 2, 4, 7, 13, 24, 44, 81, 148, 272, 499, 916, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-4} + w_{n-5} + w_{n-7} + w_{n-8}$	NEW
2	1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3}$	Tribonacci numbers, A73
5/2	1, 2, 4, 8, 15, 29, 56, 107, 206, 396, 761, 1463, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-5} + w_{n-6} + w_{n-7}$	NEW
3	1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4}$	Tetranacci numbers, A78
4	1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5}$	Pentanacci numbers, A1591
5	1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, ...	$w_n = w_{n-1} + w_{n-2} + w_{n-3} + w_{n-4} + w_{n-5} + w_{n-6}$	Hexanacci numbers, A1592

Bijective images of words avoiding some patters ?

Barcucci-Bernini-Bilotta-Pinzani patterns

In 2025, Barcucci, Bernini, Bilotta and Pinzani found a set of pattern avoiding binary words, $\mathcal{K}^{r/s}$, in bijection with q-decreasing words, for $q = r/s \in \mathbb{Q}^+$:

When $s = 1$ their result is compatible with our bijection desribed above with words avoiding 1^{r+1} !!!

Pattern avoiding and q-decreasing binary words.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
RAIRO - Theoretical Informatics and Applications, 2025

3. Bijection in the rational case

Let $q = r/s$ be an irreducible positive rational number and let

$$F^{r/s} = A^{r/s} \cup B^{r/s} \cup C^{r/s}$$

where

- $A^{r/s} = \{\, 1^r 0^t 1 \mid t = 0, \ldots, s-1 \,\}$;
- $B^{r/s} = \left\{\, 1^\ell 0^{g(\ell)} 1 \;\middle|\; \ell = 1, \ldots, r-1 \text{ and } g(\ell) = 1, \ldots, \lceil \frac{s\ell}{r} \rceil - 1 \,\right\}$;
- $C^{r/s} = \left\{\, \alpha 0 1^\ell 0^{g(\ell)} \;\middle|\; \ell = 1, \ldots, r-1 \text{ and } g(\ell) = 1, \ldots, \lceil \frac{s\ell}{r} \rceil - 1 \,\right\}$, where α is any prefix of suitable length, that is, the set of words ending by $01^\ell 0^{g(\ell)}$.

Note that, if the quantity $\lceil \frac{s\ell}{r} \rceil - 1 = 0$ for some value of ℓ, then the patterns $1^\ell 0^{g(\ell)} 1$ and $\alpha 0 1^\ell 0^{g(\ell)}$ are not consistent and they do not appear in $B^{r/s}$ and $C^{r/s}$, respectively.

Definition 3.1. We define the set $\mathcal{K}_n^{r/s}$ as

$$\mathcal{K}_n^{r/s} = \mathcal{B}_n(F^{r/s}) \setminus P^{r/s}$$

and the set $\mathcal{K}^{r/s}$ as

$$\mathcal{K}^{r/s} = \bigcup_{n \geq 0} \mathcal{K}_n^{r/s} \,,$$

where $\mathcal{B}_n(F^{r/s})$ is the set containing the n-length binary words avoiding the patterns specified in $F^{r/s}$ and

$$P^{r/s} = \left\{ 1^\ell 0^{g(\ell)} \;\middle|\; \ell = 1, \ldots, r-1 \text{ and } g(\ell) = 1, \ldots, \left\lceil \frac{s\ell}{r} \right\rceil - 1 \right\}.$$

Note that the maximal length of the words in $P^{r/s}$ is $(r-1) + \lceil \frac{s\ell}{r} \rceil - 1$, and that $P^{r/s}$ is a finite set.

Regular language ?

The words from $\mathcal{W}_q$ for positive rational q are enumerated by a rational g.f. but do not form a rational language.

The equienumerated set $\mathcal{K}^{q=r/s}$ is a regular language.

In the same paper, Barcucci, Bernini, Bilotta and Pinzani, found a regexp for this set:

$$(0+10^{\lceil s/r\rceil}+110^{\lceil 2s/r\rceil}+\ldots+1^r0^s)^*\,(\varepsilon+1+11+\ldots+1^r+1^r0+\ldots+1^r0^{s-1})$$

Barcucci-Bernini-Bilotta-Pinzani bijection

A bijection $\phi : \mathcal{K}_n^{q=r/s} \to \mathcal{W}_{q,n}$

$$\phi(b) = \begin{cases} 1^k, & \text{if } b = 1^k \text{ and } k \in [0, r]; \\ 1^{r+p}, & \text{if } b = 1^r 0^p \text{ and } p \in [1, s-1]; \\ 1^{(r+s)h+\ell+m(\ell)}, & \text{if } b = (1^r 0^s)^h 1^\ell 0^{m(\ell)},\ h \geq 1,\ \ell \in [0, r] \\ & \text{and } m(\ell) \in \left[0, (s-1)\left\lfloor \frac{\ell}{r} \right\rfloor\right]; \\ \phi(v) 0^{d+sh} 1^{\ell+rh}, & \text{if } b = (1^r 0^s)^h 1^\ell 0^d v,\ h \geq 0,\ \ell \in [0, r-1] \text{ and } d > \frac{s\ell}{r}. \end{cases}$$

Generalizes a bijection from Baril-Kirgizov-Vajnovszki (2022).

More!

Beautiful links with:

- Dyck paths of height at most 2 and avoiding certain patterns,
- integer compositions.

Dyck paths enumerated by the $\mathbb{Q}$-bonacci numbers.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
GASCom 2024, E. Proceedings in Theoretical Computer Science

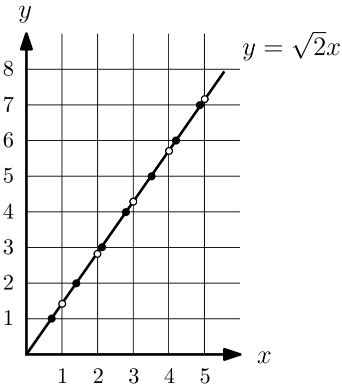
Rational Dyck paths.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
Journal of Integer Sequences, 2025.

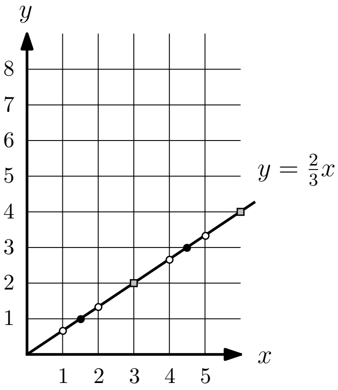
Pattern avoiding and q-decreasing binary words.
Elena Barcucci, Antonio Bernini, Stefano Bilotta, Renzo Pinzani
RAIRO - Theoretical Informatics and Applications, 2025

Real q

Sturmian words with intercept 0

Consider the line $y = q \cdot x, q \in \mathbb{R}^+$ Write 1 if the line intersects a horizontal edge, 0 in case of a vertical edge, 01 in case of a corner. The resulting infinite word is $s(q)$.





$s(\sqrt{2}) = 101011010110...$

$s\left(\frac{2}{3}\right) = 0100101001...$

From Sturmian to q-decreasing

E.g., slope is $q = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}}$

Sturmian word $s(1/\varphi) = 0100101001001010...$ *(aka Fibonacci word)*

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1. Construct q-suffixes from sturmian prefixes ending with 1

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$$01 \rightarrow \widehat{001}$$

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$$01001 \rightarrow \widehat{000011}$$

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$$\cdots$$

2. Allow $\widehat{0}$ as q-suffix.

From Sturmian to q-decreasing

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Length 0: one empty word

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Length 1: 1,

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Length 0: one empty word
Length 1: 1, $\widehat{0}$

From Sturmian to q-decreasing

E.g., slope is $q = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}}$

Sturmian word $s(1/\varphi) = 01001010010010 10\ldots$ *(aka Fibonacci word)*

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$$01001 \to \widehat{000011}$$
$$\ldots$$

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Length 0: one empty word
Length 1: $1, \widehat{0}$
Length 2: $11, 1\widehat{0}, \widehat{0}\widehat{0}$

From Sturmian to q-decreasing

E.g., slope is $q = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}}$

Sturmian word $s(1/\varphi) = 01001\,01001001010\ldots$ *(aka Fibonacci word)*

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Length 0: one empty word
Length 1: $1, \widehat{0}$
Length 2: $11, 1\widehat{0}, \widehat{0}\widehat{0}$
Length 3: $111, 11\widehat{0}, 1\widehat{0}\widehat{0}, \widehat{001}, \widehat{0}\widehat{0}\widehat{0}$

From Sturmian to q-decreasing

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$$\dots$$

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Length 0: one empty word

Length 1: $1, \widehat{0}$

Length 2: $11, 1\widehat{0}, \widehat{0}\widehat{0}$

Length 3: $111, 11\widehat{0}, 1\widehat{0}\widehat{0}, \widehat{001}, \widehat{0}\widehat{0}\widehat{0}$

...

Length 24: $1111\widehat{0}\widehat{0}\widehat{0}\widehat{000011}\widehat{001}\widehat{0}\widehat{0}\widehat{000011}$, ...

From Sturmian to q-decreasing

E.g., slope is $q = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}}$

Sturmian word $s(1/\varphi) =$ 01001 01001001010... *(aka Fibonacci word)*

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$$01001 \rightarrow \widehat{000011}$$
$$\dots$$

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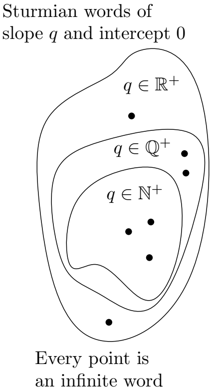
3. Construct q-decreasing words as a sequence of 1s followed by a sequence of q-suffixes.

Length 0: one empty word
Length 1: $1, \widehat{0}$
Length 2: $11, 1\widehat{0}, \widehat{0}\widehat{0}$
Length 3: $111, 11\widehat{0}, 1\widehat{0}\widehat{0}, \widehat{001}, \widehat{0}\widehat{0}\widehat{0}$
...
Length 24: $1111\widehat{0}\widehat{0}\widehat{0}\widehat{000011}\widehat{001}\widehat{0}\widehat{0}\widehat{000011}$, ...

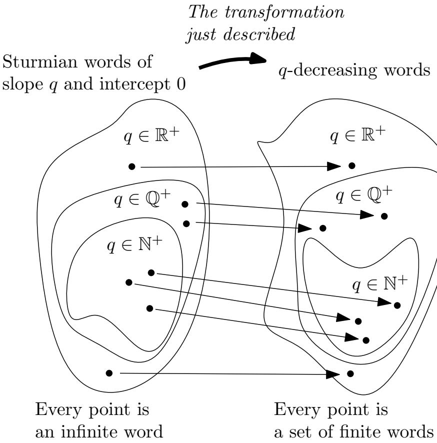
Cards.: 1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, 182, 286, 448, ...

General picture



The picture is drawn in collaboration with Sima, Fedia and Vasia.

General picture



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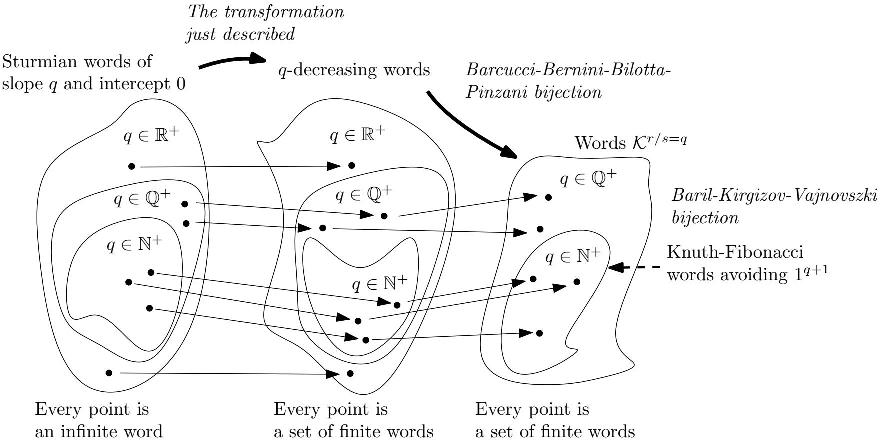
General picture



The picture is drawn in collaboration with Sima, Fedia and Vasia.

How east does $|\mathcal{W}_{q,n}|$ grow ?

Consider the following function

$$\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$

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For $q = 1$, we get the golden ratio ($\mathcal{W}_{1,n}$ is counted with the Fibonacci numbers).

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For $q = 2$, it is the tribonacci constant.

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For $q = 5/3$?

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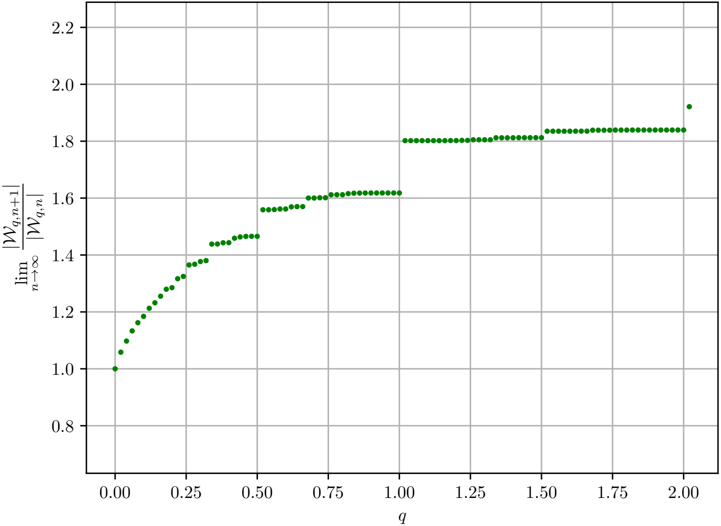
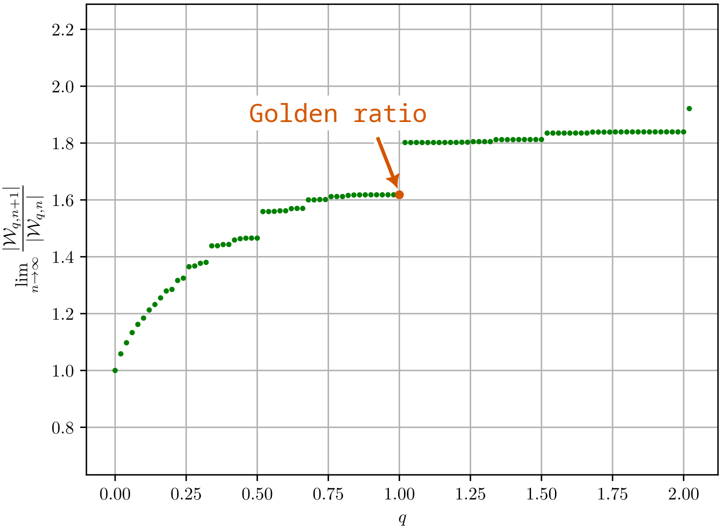
($\mathcal{W}_{1,n}$ is counted with the Fibonacci numbers).
For $q = 2$, it is the tribonacci constant.
For $q = 5/3$?
For $q = \varphi$?

GeneℝaIization of the golden ratio



$$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$$ as a function of q.

GeneℝaIization of the golden ratio



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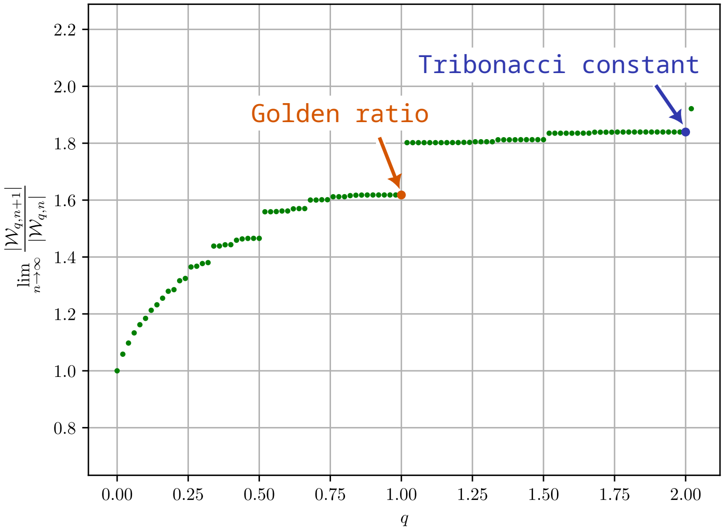
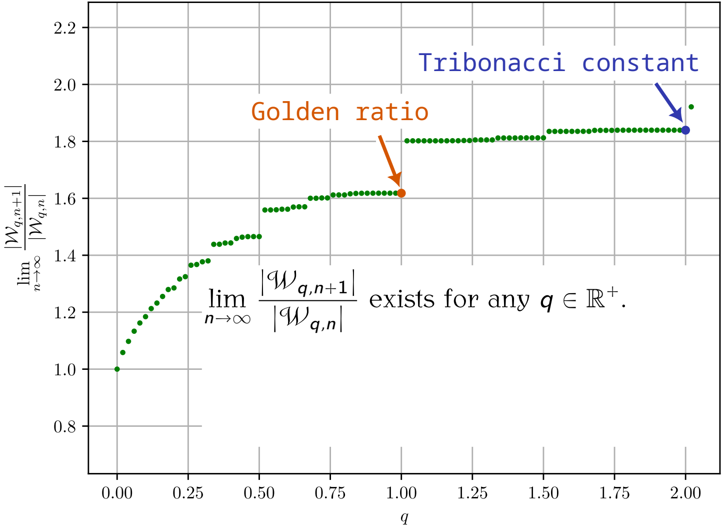


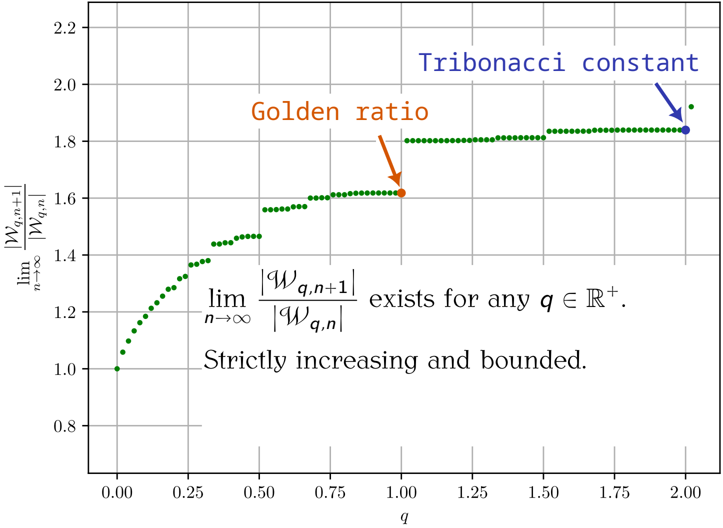
$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

Gene$\mathbb{R}$alization of the golden ratio



$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

Gene$\mathbb{R}$alization of the golden ratio

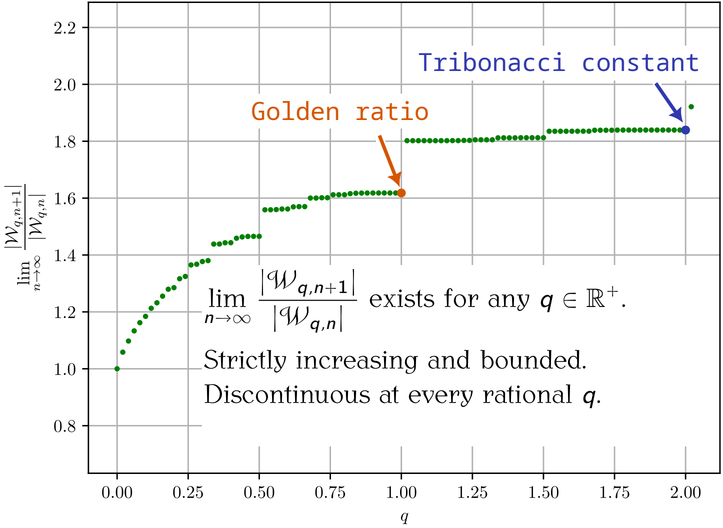


$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ exists for any $q \in \mathbb{R}^+$.

Strictly increasing and bounded.

$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ as a function of q.

Gene$\mathbb{R}$alization of the golden ratio



$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ exists for any $q \in \mathbb{R}^+$.

Strictly increasing and bounded.
Discontinuous at every rational q.

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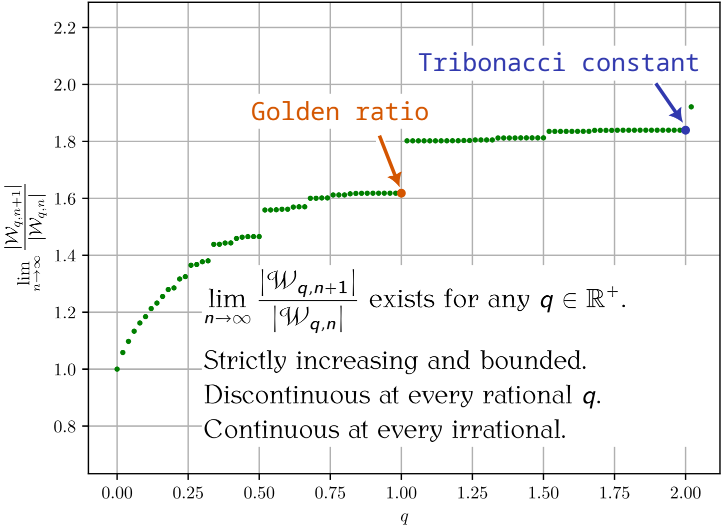

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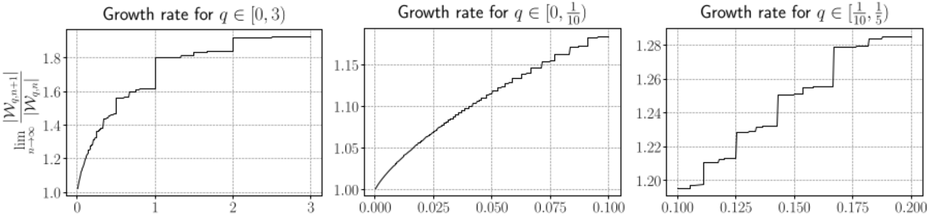


Growth rate for $q \in [0, 3)$
$\lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$
1.0
1.2
1.4
1.6
1.8
0
1
2
3
Growth rate for $q \in [0, \frac{1}{10})$
1.00
1.05
1.10
1.15
0.000
0.025
0.050
0.075
0.100
Growth rate for $q \in [\frac{1}{10}, \frac{1}{5})$
1.20
1.22
1.24
1.26
1.28
0.100
0.125
0.150
0.175
0.200

General case

$$1 \ldots 1\, \sigma_1\, \sigma_2\, \ldots\, \sigma_\ell$$

A word from $\mathcal{W}_q$ is a sequence of 1s followed by a sequence of admissible suffixes of the form

$$0^{1+\lfloor i/q \rfloor} 1^i,$$

where $i \geq 0$.

The generating function is

$$W_q(x) = \frac{1}{(1-x)\left(1 - \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)}.$$

Theorem (Dovgal-Kirgizov, 2025)

For any real $q > 0$, the number of q-decreasing words of length n grows as $C_q \cdot \Phi(q)^n$, where $1/\Phi(q)$ is the unique smallest in modulus root of $1 - \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}$, and

$$C_q = -\frac{\Phi(q)}{\left((1-x)\left(1 - \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)\right)' \left(1/\Phi(q)\right)}.$$

Another decomposition

The equation $1 = \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}$ shares the smallest in modulus root with

$$A_q := 1 - \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor + j}.$$

A set $\mathcal{F}_q$ of factors $0^a 1^b$ such that $qa > b$ and $b \geq 1$.

Any word $w \in \mathcal{W}_q$ as a sequence of 1s followed by a sequence of factors from $\mathcal{F}_q$, followed by a sequence of 0s. Any of these sequences can be empty.

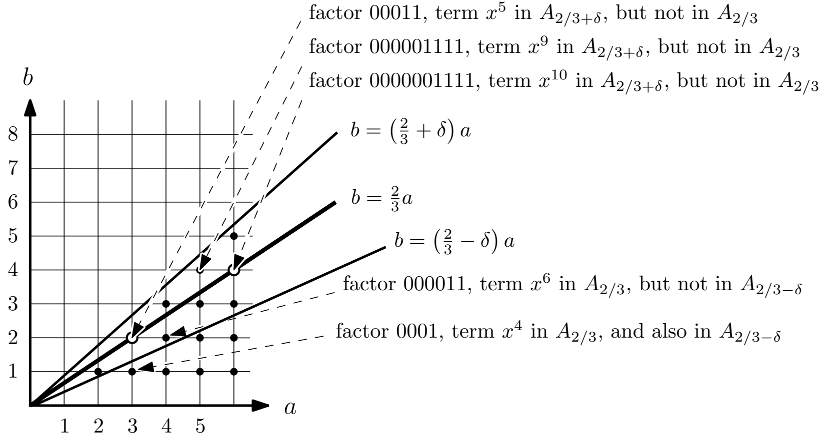
$$w = \overbrace{1 \ldots 1}^{\text{some ones}} \; \overbrace{f_1 f_2 \ldots . f_k}^{f_\ell \in \mathcal{F}_q} \; \overbrace{0 \ldots 0}^{\text{some zeros}}, \quad \text{where } \mathcal{F}_q = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{\infty} \{ \overbrace{0 \ldots 00}^{1+\left\lfloor \frac{i}{q} \right\rfloor + j \text{ zeros}} \underbrace{1 \ldots 11}_{i \text{ ones}} \}.$$

G.f. is

$$W_q(x) = \frac{1}{1-x} \cdot \frac{1}{A_q} \cdot \frac{1}{1-x}.$$

$$A_q := 1 - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor + j}.$$

Consider the grid $\mathbb{Z}^+ \times \mathbb{Z}^+$, and make every point (a, b) correspond to a factor 0^a1^b, where $qa > b$. The sum in A_q correspond to all points with positive integer coordinates found under the line $b = qa$.

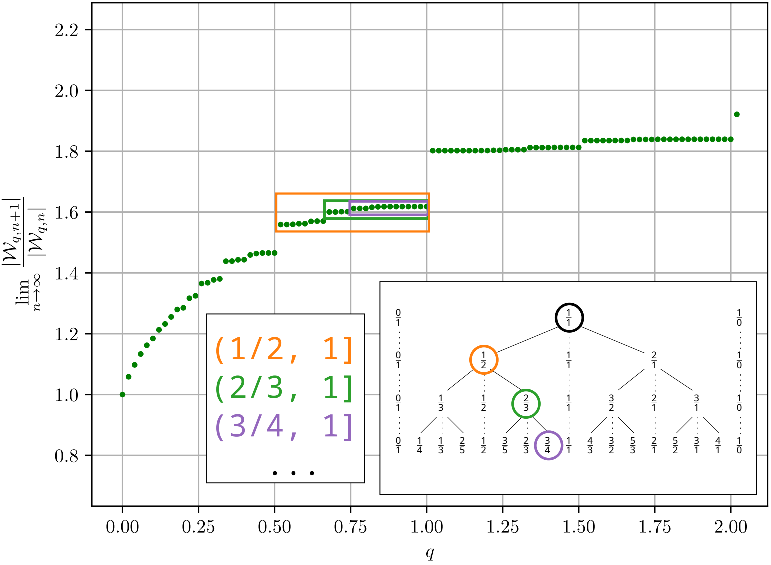


Theorem (Dovgal-Kirgizov 2025)

The function $\Phi(q) = \lim_{n\to\infty} \frac{|\mathcal{W}_{q,n+1}|}{|\mathcal{W}_{q,n}|}$ *is*

- **a**) *strictly increasing over* $q \in [0, \infty)$*;*
- **b**) *bounded,* $1 \leq \Phi(q) < 2$*, with* $\Phi(0) = 1$ *and* $\lim_{q\to\infty} \Phi(q) = 2$*;*
- **c**) *left-continuous (and right-discontinuous) at every positive rational point;*
- **d**) *continuous at every positive irrational point.*

Fractal, Stern-Brocot tree and Minkowski's $?(x)$



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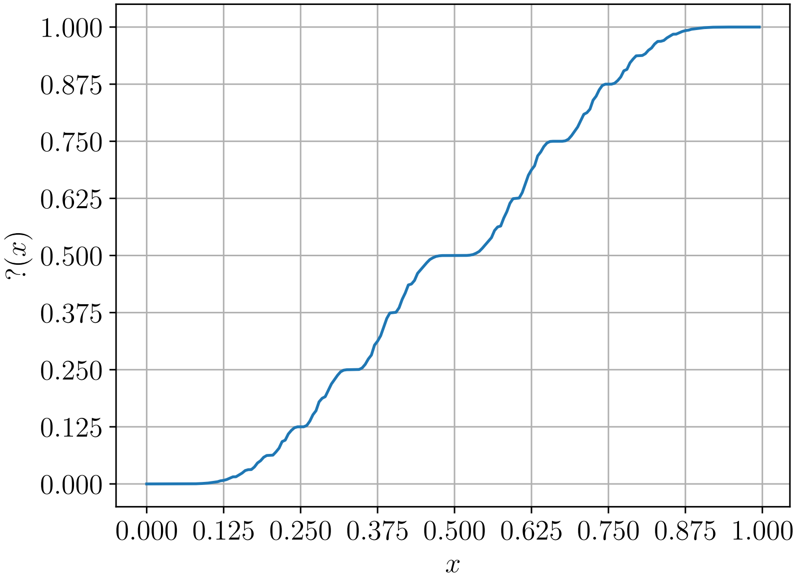

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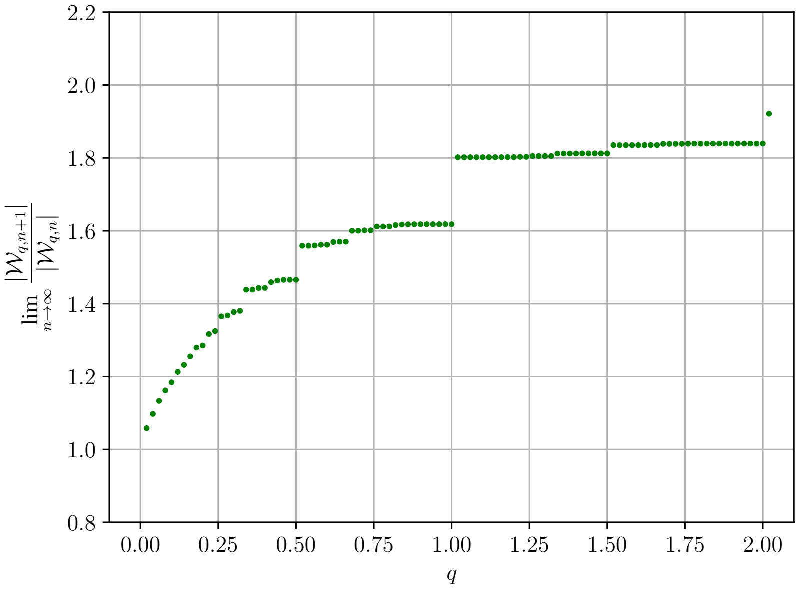
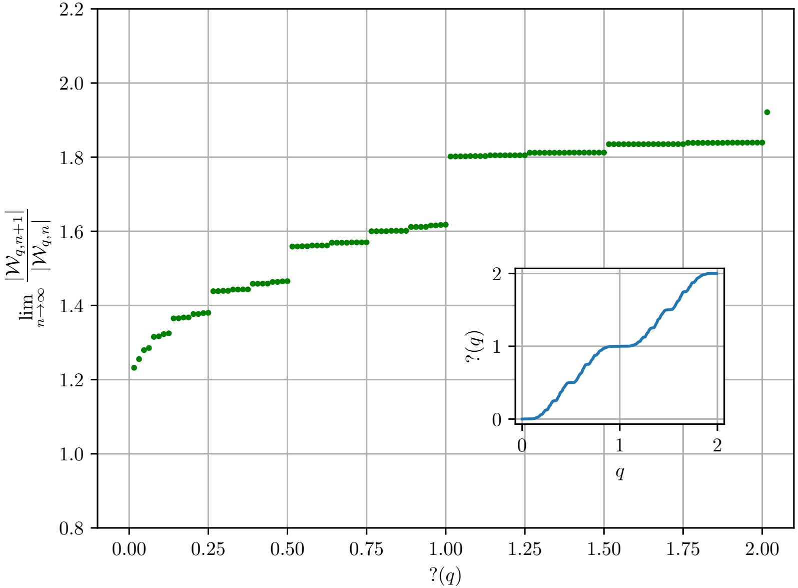


Fractal, Stern-Brocot tree and Minkowski's $?(x)$



q-decreasing words with rational and real q

- Q-bonacci words and numbers. Sk, Fibonacci conference
- The Fibonacci Quarterly, 2022
  https://kirgizov.link/talks/fiboconf.pdf
  https://arxiv.org/abs/2201.00782

- Sturm meets Fibonacci in Minkowski's fractal bar.
  Sergey Dovgal and Sk
  Permutation Patterns 2023
  https://kirgizov.link/publications/pp23.pdf

- Structure and growth of $\mathbb{R}$-bonacci words.
  Sergey Dovgal and Sk
  Electronic Journal of Combinatorics, 2025
  http://arxiv.org/abs/2310.01213

Gray codes.

In a *k-Gray code* consecutive words differs in at most k positions.

Classical Gray code yields 3-Gray code for $\mathcal{W}_q$

Vajnovszki in "Gray code order for Lyndon words" (2007) introduces the notion of *absorbent set*.

Definition.
A binary word set $\mathcal{X} \subset \{0,1\}^n$ *is called* absorbent *if for any* $u \in \mathcal{X}$ *and any* k, $1 \leq k < n$, $u_1 u_2 \ldots u_k 0^{n-k}$ *is also a word in* $\mathcal{X}$.
(Note that we can "mirror" this definition, adding 0s at the left.)

Lemma.
Any absorbent set listed in binary reflected Gray code order yields a 3-Gray code.

Corollary.
The set of n-length q-decreasing words admits a 3-Gray code.

3-Gray code, efficient generation

Efficient generation of q-decreasing words in lexicographical 3-Gray code order, $O(1)$ per word, is possible. It satisfies Frank Ruskey's constant amortized time (CAT) principle.

CAT principle for a recursive generation algorithm:

1. Every call results in the output of at least one object;
2. Excluding the computation done by recursive calls the amount of computation of any call is proportional to the degree of the call (that is, the number of subsequent recursive calls produced by the call);
3. The number of calls of degree one is $O(N)$, where N is the number of generated objects.

[Roelants van Baronaigien and Ruskey, 1993]

3-Gray code, efficient generation

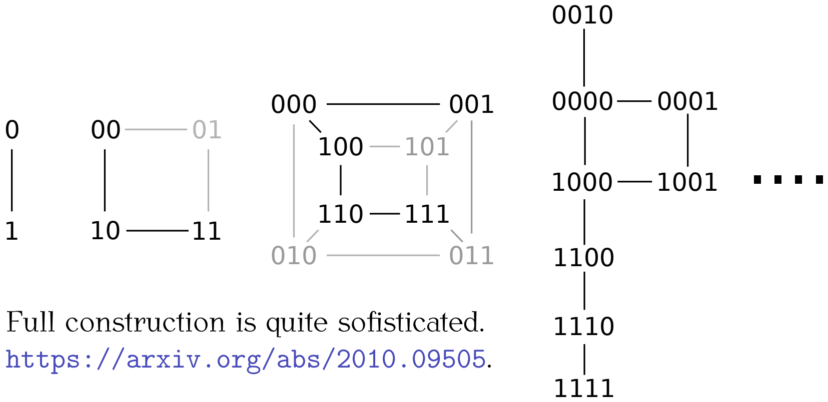
```
procedure LexFib (pos, delta: integer)
  if (pos = n + 1) print w;
  else w[pos] := 0;
        if (w[pos − 1] = 1) d := q − 1; else d:=delta + q; endif
        LexFib(pos + 1, d);
        if (delta > 0)
           w[pos] := 1; LexFib(pos + 1, delta − 1);
        endif
  endif
end procedure
```

It generates q-decreasing words in the lexicographical order.

$w[0]$ is initialized by 1 and the parameter *delta* is the number of consecutive 1s that can be added to the current generated prefix without violating the q-decreasingness. Main call is `LexFib(1, n)`.

1-Gray code and solved conjecture

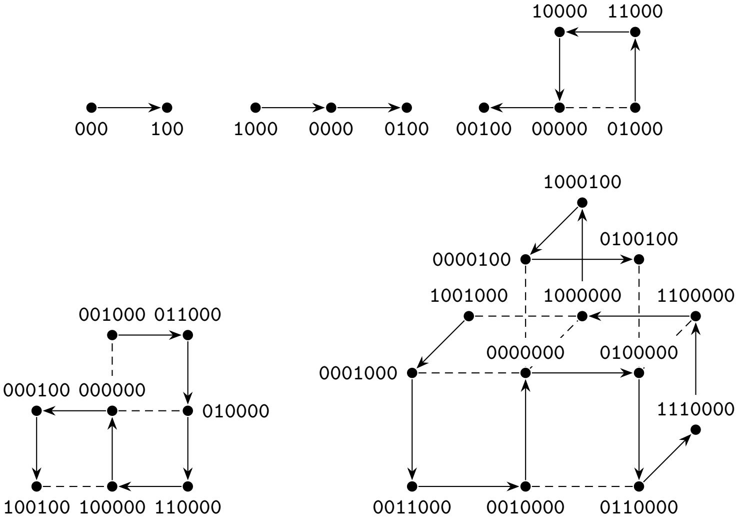
For $q = 1$ we managed to construct a 1-Gray code.



Full construction is quite sofisticated.
https://arxiv.org/abs/2010.09505.

We solve the conjecture about the existence of a Hamiltonian path in Fibonacci-run graphs, see Ömer Eğecioğlu and Vesna Iršič, "Fibonacci-run graphs I: Basic properties", 2021.

Eğecioğlu-Iršič graphs



Fibonacci-run graphs for small values of n. Vertices correspond to the reverse of words from $\mathcal{W}_n^1$ beginning with 0.

Wong-Liu-Lam-Im 2-Gray code

Algorithm 1 Recursive algorithm to generate $\mathcal{R}_q^n$ for q-run constrained sequences.

function QRUN(n, q)
  $R \leftarrow []$
  $p \leftarrow 1$
  for i from 0 to $n - \lfloor \frac{1}{q} + 2 \rfloor$ **do**
    if i is even **then**
      for j from 1 to $n - i - \left\lfloor \frac{n-i}{q+1} + 1 \right\rfloor$ **do**
        $R \leftarrow \{R, 0^{n-i-j}1^j \cdot \text{QRUN}(i, q)^{-p}\}$
        $b_1 b_2 \cdots b_n \leftarrow$ last string in R
        $p \leftarrow 1 - b_n$
    else
      for j from $n - i - \left\lfloor \frac{n-i}{q+1} + 1 \right\rfloor$ to 1 **do**
        $R \leftarrow \{R, 0^{n-i-j}1^j \cdot \text{QRUN}(i, q)^{-p}\}$
        $b_1 b_2 \cdots b_n \leftarrow$ last string in R
        $p \leftarrow 1 - b_n$
  $R \leftarrow \{R, 0^n\}$
  return R

- Generating cyclic 2-Gray codes for Fibonacci q-decreasing words.
  Dennis Wong, Bowie Liu, Chan-Tong Lam and Marcus Im
  WALCOM: Algorithms and Computation, 2024.

- Generating a cyclic 2-Gray code for Lucas words in constant amortized time.
  Liu, Wong, Lam, and Im.
  CPM 2025

Wong-Liu-Lam-Im 2-Gray code

Algorithm 2 A simple algorithm to generate $\mathcal{Q}_q^n$ for q-decreasing sequences.

function QDEC(n, q)
  $Q \leftarrow []$
  $p \leftarrow 1$
  $r \leftarrow n$
  while $r \geq 0$ **do**
    $Q \leftarrow \{Q, 1^r \cdot \text{QRUN}(n - r, q)^{-p}\}$
    $b_1 b_2 \cdots b_n \leftarrow$ last string in Q
    $p \leftarrow 1 - b_n$
    $r \leftarrow r - 2$
  $Q' \leftarrow []$
  $p \leftarrow 1$
  $r \leftarrow n - 1$
  while $r \geq 0$ **do**
    $Q' \leftarrow \{Q', 1^r \cdot \text{QRUN}(n - r, q)^{-p}\}$
    $b_1 b_2 \cdots b_n \leftarrow$ last string in Q'
    $p \leftarrow 1 - b_n$
    $r \leftarrow r - 2$
  $Q \leftarrow \{Q, \overline{Q'}\}$
  return Q

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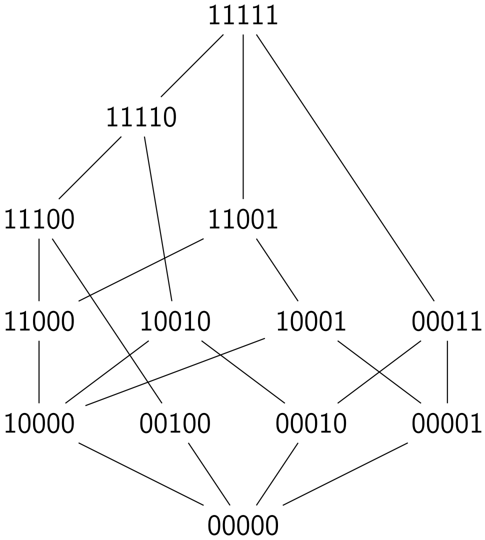
Lattice *come dessert*

and other structures

Lattice (under componentwise order)



The lattice $\mathbb{W}_5^1$.
It contains 20 coverings (edges), 7 meet-irreducible elements, 5 join-irreducible elements and 56 intervals.

Enumeration in the lattice of q-decreasing words. Jean-Luc Baril, Nathanaël Hassler and Sk. 2025, https://arxiv.org/abs/2511.09480

Hassler-Vajnovski-Wong words and codes

Length n binary words having k 1s with the property that any prefix contains at least p times as many 0s as 1s.

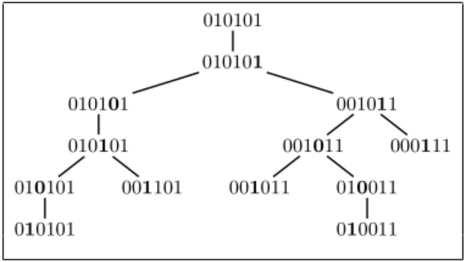


FIGURE 3. The tree of recursive calls produced by the call of PREF$(6, 3)$ with $p = 1$, when the initial word is $b = 010101$. A homogeneous Gray code for $C_1(6, 3)$ is obtained by collecting the words b at the leaves.

✎ Greedy gray codes for some restricted classes of binary words.
Nathanaël Hassler, Vincent Vajnovszki and Dennis Wong
GASCom 2024, E. Proceedings in Theoretical Computer Science

Even more!

- Fibonacci Cubes with Applications and Variations.
  Ömer Eğecioğlu, Sandi Klavžar and Michel Mollard
  World Scientific, 2023

- Combinatorial Gray codes-an updated survey, Torsten Mütze
  https://arxiv.org/pdf/2202.01280.pdf
  Electronic Journal of Combinatorics, 2022

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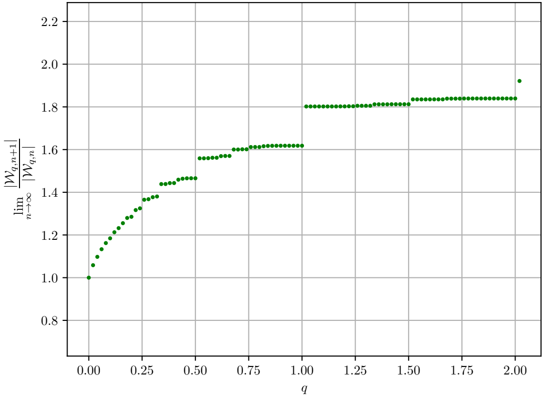
- Explain the order of jumps...

Explain the order of jumps



Here is the sequence of positive rational numbers ordered by corresponding jumps of the function $\Phi(q) = \lim_{n\to\infty} |\mathcal{W}_{q,n+1}|/|\mathcal{W}_{q,n}|$

$$1, \frac{1}{2}, 2, \frac{1}{3}, \frac{1}{4}, 3, \frac{2}{3}, \frac{1}{5}, \frac{1}{6}, \frac{3}{2}, \frac{1}{7}, 4, \frac{2}{5}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{3}{4}, \frac{1}{11}, \frac{2}{7}, \frac{1}{12}, 5, \frac{3}{5}, \frac{1}{13}, \frac{4}{3}, \frac{1}{14}, \frac{2}{9}, \frac{1}{15}, \frac{1}{16}, ?$$

Next term is ... ???

Mille volte grazie !