Packing coloring and subsets preserving path distance

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Packing coloring

Graph $G = (V, E)$
Usual distance between vertices $d : V \times V \to \mathbb{N}$

Packing coloring

Function $c_\rho : V \to \mathbb{N}$ such that $\forall u \neq v \in V$ we have

$$c_\rho(u) = c_\rho(v) = i \quad \Rightarrow \quad d(u, v) > i$$

$\chi_\rho(G)$ — minimal number of colors.

Historical notes

- **Broadcast Chromatic Numbers of Graphs**
  [Goddard, Hedetniemi, Hedetniemi, Harris, Rall, 2003]
- **An eccentric coloring of trees**
  [Sloper, 2004]
- **On the packing chromatic number of Cartesian products, hexagonal lattice, and trees**
  [Brešara, Klavžarb, Rall, 2007]

Some complexity results

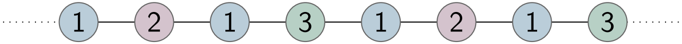
- Deciding whether $\chi_\rho(G) \geq 4$ is NP-complete
  [Goddar et al.]
- Given a tree T and $k \in \mathbb{N}$ deciding whether $\chi_\rho(T) \geq k$ is NP-complete.
  [Fiala, Golovach, 2010]

Historical notes 2

Partitionnement, recouvrement et colorabilité dans les graphes

Nicolas Gastineau, Ph.D. thesis, 2014

Infinite path



[Goddard, Hedetniemi, Hedetniemi, Harris, Rall, 2003]

Infinite square lattice

The packing chromatic number of the square lattice is at least 12

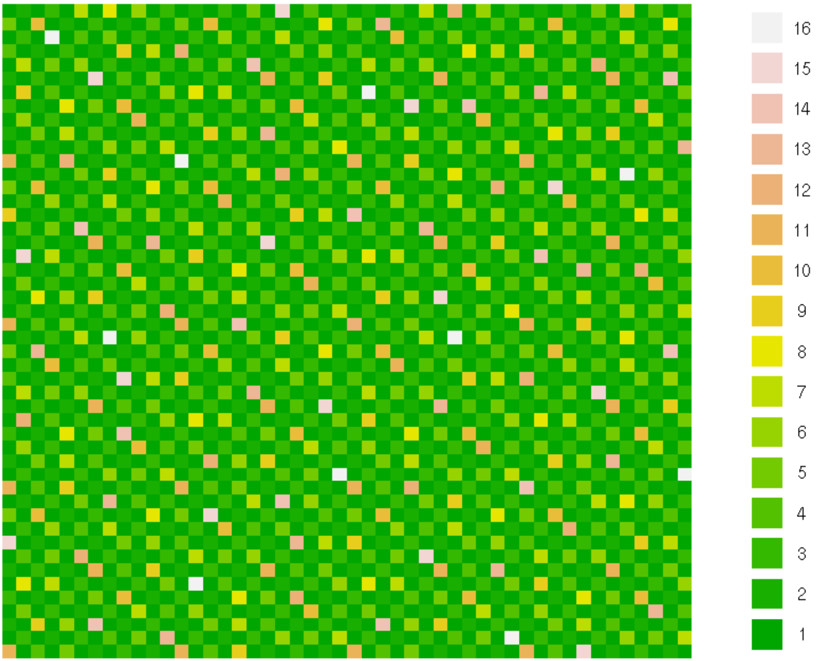
[Ekstein, Fiala, Holub, Lidický, 2011]

The packing chromatic number of the infinite square lattice is less than or equal to 15

"Using a SAT-solver on top of a partial previously-known solution...."

[Martin, Raimondi, Chen, Martin, 2015]

- 48×48 periodic solution with 16 colors (**next slide...**)
- 72×72 periodic solution with 15 colors


16
15
14
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2
1

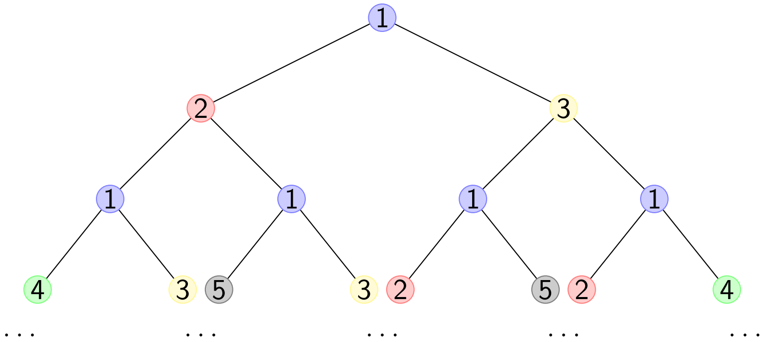
Infinite trees

T_k is an infinite k-ary tree.

Known results

- $\chi_\rho(T_2) \leq 7$
- $\chi_\rho(T_k) = \infty$ if $k \geq 3$

[Sloper, 2004]



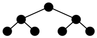
Our first result

Theorem

$\chi_\rho(T_2) = 7$

Proof.

- We known that $\chi_\rho(T_2) \leq 7$ [Sloper]
- We show $\chi_\rho(T_2) > 6$ by brute-force over increasing sequence of induced subtrees

k	$T_{2,k}$	$\chi_\rho(T_{2,k})$
0	•	1
1		2
2		3
3	...	3
4	...	4
5	...	5
6	...	6
7	...	7

Starting from color k

Starting from color k

Another variant of packing coloring appears when we authorise only colors greater than k.

Packing coloring 2

Function $c_\rho : V \to \mathbb{N}$ such that $\forall u \neq v \in V$ we have

$$c_\rho(u) = c_\rho(v) = i \quad \Rightarrow \quad d(u,v) > i \text{ and}$$

$$c_\rho(\cdot) \geq k$$

$\chi_\rho^k(G)$ — minimal number of colors.

Asymptotically packing colorable

For any k we have $\chi_\rho^k(G) \leq f(k) < \infty$.
$f(k)$ is a packing function.

Theorem

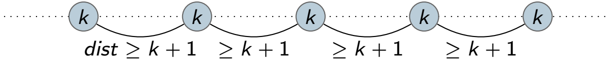
[Goddard, Hedetniemi, Hedetniemi, Harris, Rall, 2003]

1. $\chi_\rho^k(P_\infty) \leq 3k+2$ *if* $k < 34$
2. $\chi_\rho^k(P_\infty) \leq 3k-1$ *if* $k \geq 34$

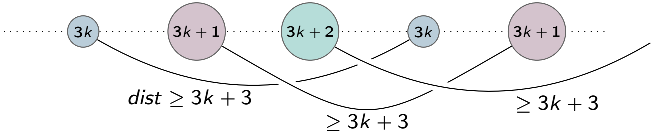
Proof.

1. By computer search.
2. By induction. The base case $\chi_\rho^k(P_\infty) \leq 3k-1$ for $k = 34$ is obtained by computer-based greedy coloration.

$\chi_\rho^k(P_\infty) \leq 3k-1$



$\chi_\rho^{k+1}(P_\infty) \leq 3(k+1)-1$



Infinite square (and also hexagonal) lattice L is not asymptotically packing colorable, since $\chi^2_\rho(L) = \infty$

Proof's idea

Define by $d(X_k)$ the proportion of a maximal subset that can be colored by a color k. Consider an optimistic scenario and show that $\sum_{i=2}^{\infty} d(X_k) < 1$.

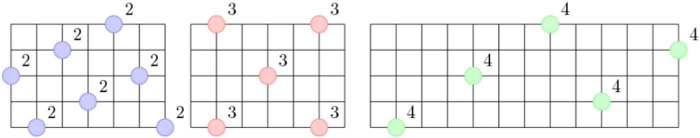


Figure 4: The sets X_2 (2-packing), X_3 (3-packing) and X_4 (4-packing) in $\mathbb{Z}^2$.

[Gastineau, Kheddouci, Togni, 2015]

Infinite trees are not asymptotically packing colorable

T_k is an infinite k-ary tree.

Known results

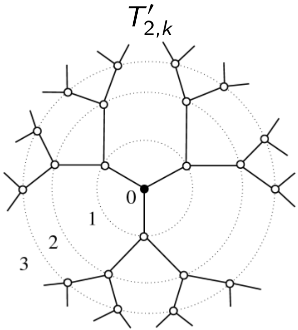
- $\chi_\rho(T_k) = \chi_\rho^1(T_k) = \infty$ if $k \geq 3$ [Sloper, 2004]

Theorem

Infinite binary tree is not asymptotically packing colorable

Proof. On the next slides...

Binary tree is not asymptotically packing colorable

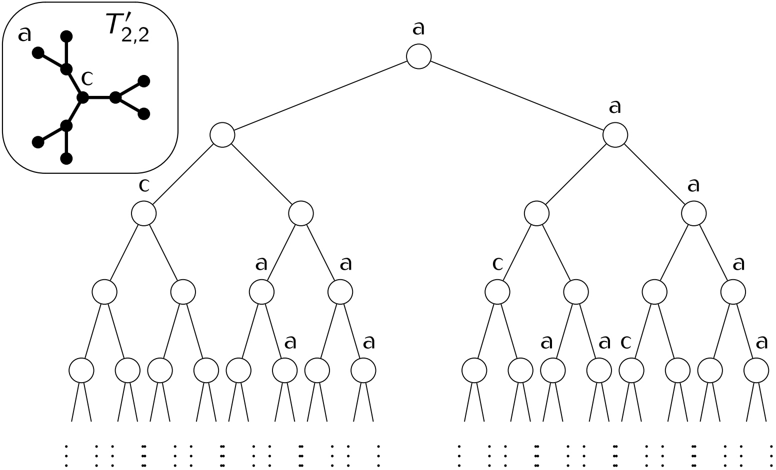


(figure by Bela Mulder)

Denote by $T'_{2,k}$ a Bethe lattice (3-Cayley tree)

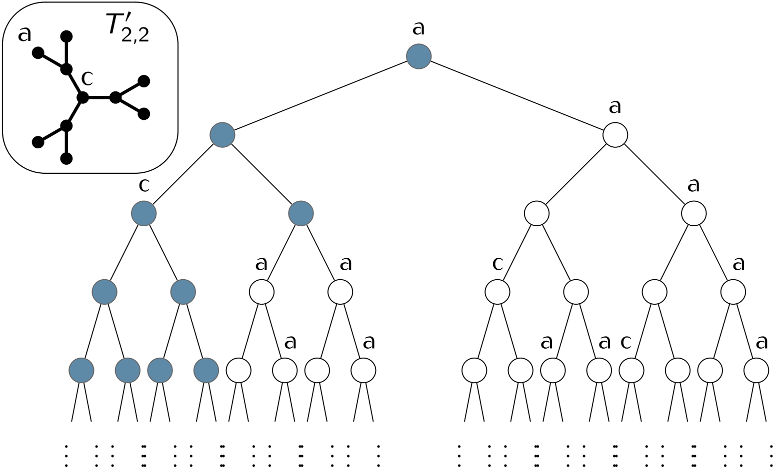
1. $|T'_{2,n}| = 3 \cdot 2^n - 2$
2. $diam(T'_{2,n}) = 2n$
3. T_2 can be covered by several induced $T'_{2,n}$

Binary tree covered by Bethe lattices



Colors represent Bethe lattices trees with 3 levels.

Binary tree covered by Bethe lattices



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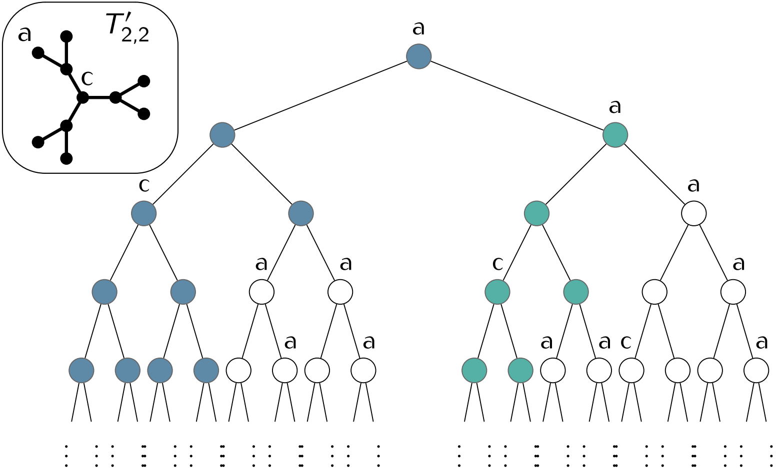

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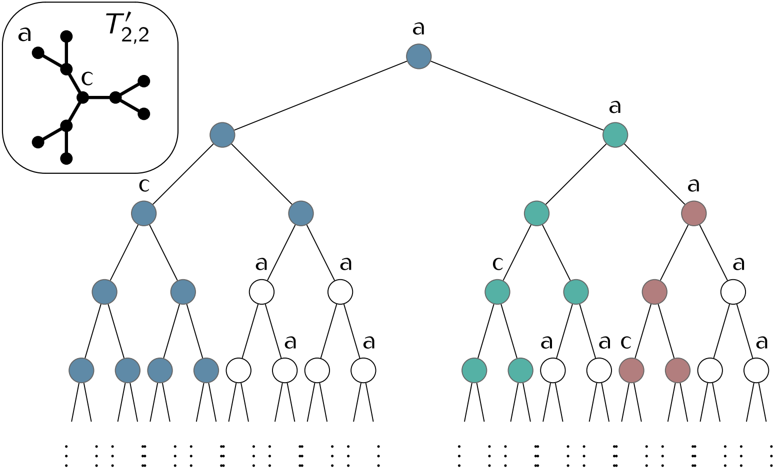

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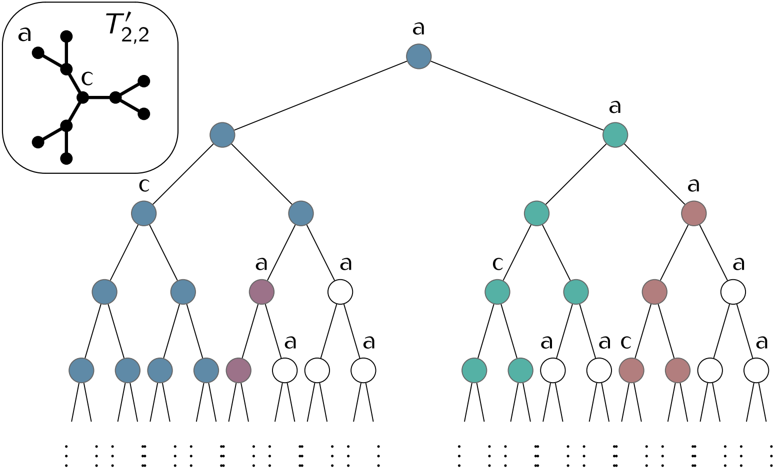

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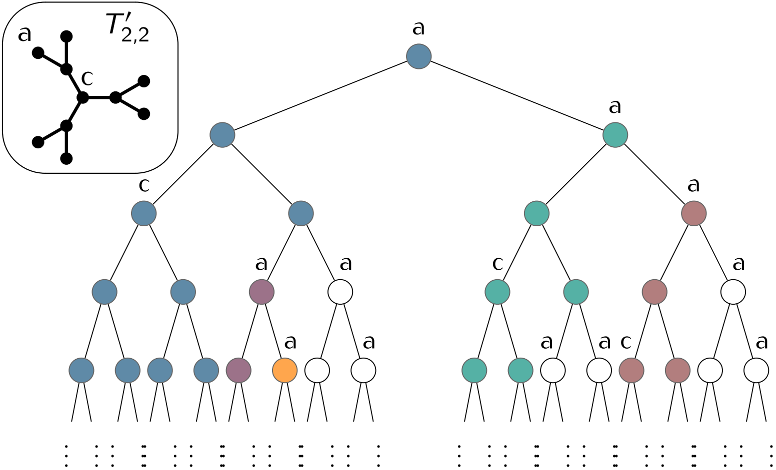

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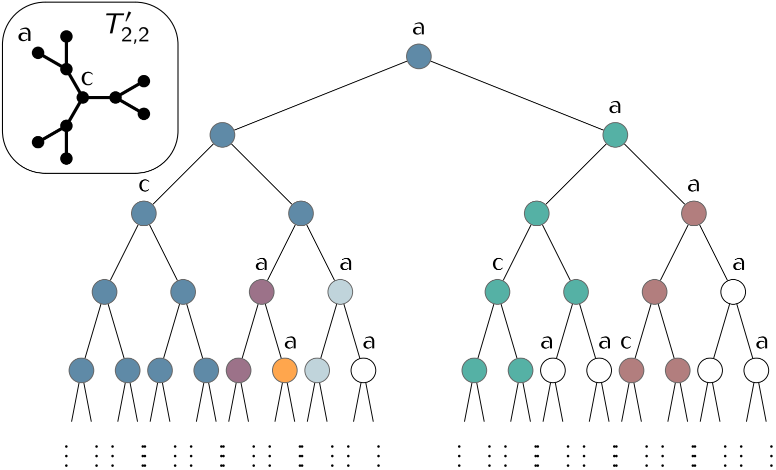

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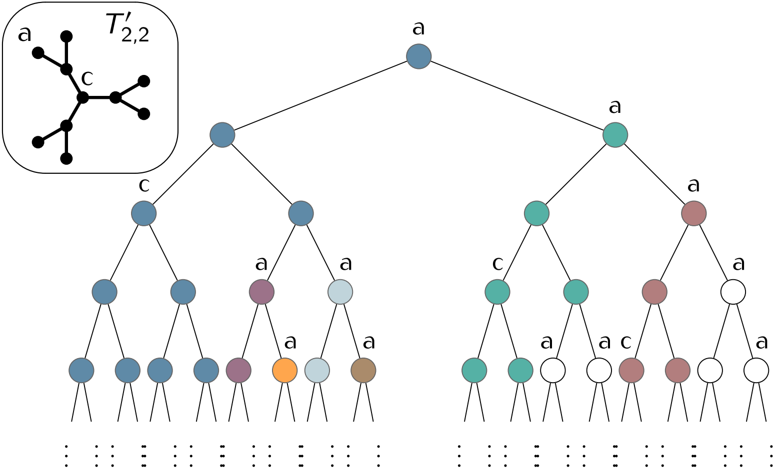

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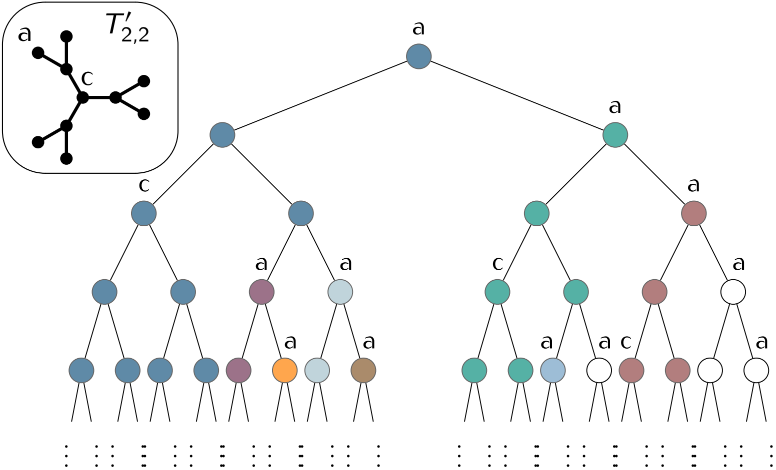

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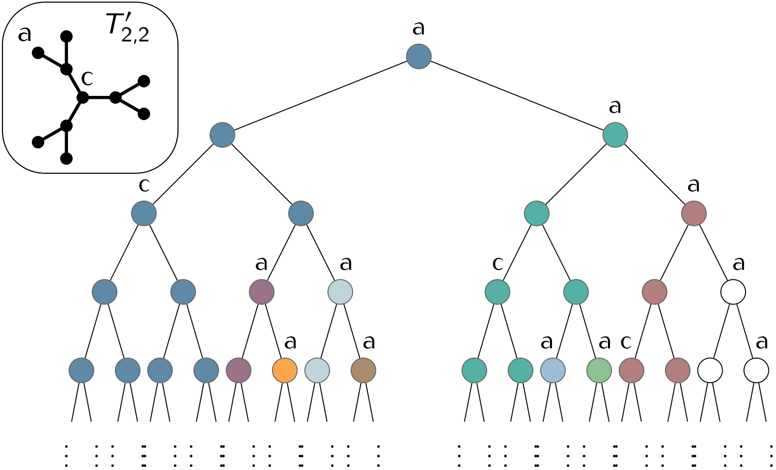

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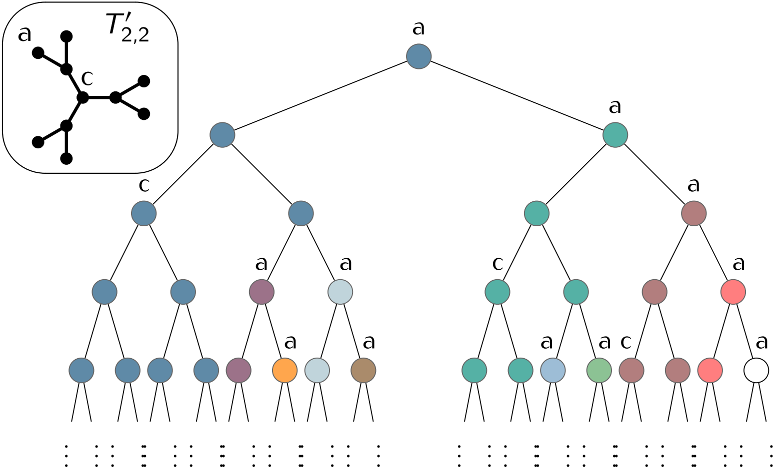
Binary tree covered by Bethe lattices



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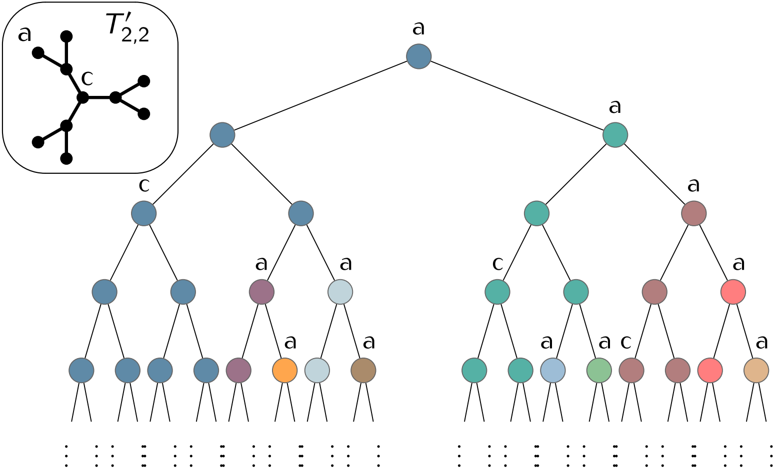
Binary tree covered by Bethe lattices



Colors represent Bethe lattices trees with 3 levels.

Binary tree is not asymptotically packing colorable. 2

1. T_2 can be covered by several induced $T'_{2,n}$
2. $diam(T'_{2,n}) = 2n$
3. $|T'_{2,n}| = 3 \cdot 2^n - 2$

- At most $\frac{1}{3 \cdot 2^n - 2}$ vertices can be colored with color $2n$ (or with color $2n+1$).
- colors 2,4,... $\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n - 2} < 0.44$
- colors 3,5,... $\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n - 2} < 0.44$
- colors 2,3,5,6 ... $2\sum_{n=1}^{\infty} \frac{1}{3 \cdot 2^n - 2} < 0.88$

Thus, in optimistic scenario the sum of density of all colored vertices is less than 1. It means, that we cannot color using only these colors.

Expanding maps and packing coloration

(Eventually) expanding maps

H, G are two metric spaces (graphs equipped with standard distance, for example).

$\phi : H \to G$ is a expanding map if $d_H(u, v) \leq d_G\big(\phi(u), \phi(v)\big)$

Immediately we have

A packing coloration of H is also a packing coloration of $\phi(H) \subset G$

Path (distance preserving) decomposition

Path distance preserving subset

If $\psi : P_\infty \to G$ is a expanding map from an infinite path to a graph G, then $\psi(P_\infty)$ *preserves a path distance*.

- We know that P_∞ is asymptotically packing colorable, moreover $\chi_\rho^k(P_\infty) \leq 3k + 2$ for any k.

Path (distance preserving) decomposition

Path distance preserving subset

If $\psi : P_\infty \to G$ is a expanding map from an infinite path to a graph G, then $\psi(P_\infty)$ *preserves a path distance.*

- We know that P_∞ is asymptotically packing colorable, moreover $\chi_\rho^k(P_\infty) \leq 3k+2$ for any k.

Theorem

If the set of vertices of G is a finite disjoint union of subsets that preserve a path distance

$$G = \biguplus_{i \in I, |I| < \infty} \psi_i(P_\infty)$$

then G is asymptotically packing colorable, moreover

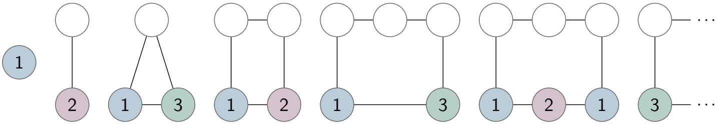
$$\chi_\rho^k(G) \leq f_{|I|}(k)$$

where $f_1(k) = 3k+2$ and $f_n = f\left(f_{n-1}(k)+1\right)$.

Decomposition on paths

An example

An example $C_* = \bigcup_{n\in\mathbb{N}} C_n$



- we color lower path by 12131213...
- we color upper path by $3 \cdot 4 + 2$ colors starting from the color 4.

$$\chi_\rho(C_*) \leq 3 + 3 \cdot 4 + 2 \leq 17$$

Consequences, additional results

- Infinite k-ary tree ($k \geq 2$) is **not decomposable** into finite number of infinite paths.
- Square and hexagonal lattices are **not decomposable** into finite number of infinite paths.

- If a graph can be decomposed, the minimal number of paths is an interesting parameter, denoted by $p_\infty(G)$.
  - $p_\infty\left(\bigcup_{n\in\mathbb{N}} C_n\right) = 2$
  - Let S_k be an infinite star with k edges, $p_\infty(S_k) = \lceil \frac{k}{2} \rceil$
  - Let P_∞^k be a graph with vertex set $\mathbb{Z}$ and edge set $\{ab \mid |a-b| \leq k\}$, we have $p_\infty(P_\infty^k) \leq k$

Conclusion

1. Packing coloring number of binary tree is 7.
2. Binary tree is not asymptotically packing colorable.
3. Path decomposition into finite number of paths implies asymptotically packing colorability.
4. Minimal number of paths in path decomposition is an interesting parameter.
5. **Conjecture: converse of 3 for connected graphs**

Conjectures

1. Does the asymptotic packing colorability of a connected graph implies the existence of decomposition into finite number of paths ?
2. Is there a graph G such that $\chi_\rho^1(G) < \infty$, $\chi_\rho^2(G) < \infty$ but $\chi_\rho^3(G) = \infty$?
3. Does $\chi_\rho^2(G) = \infty$ imply $\chi_\rho^k(G) = \infty$ for any $k > 2$
4. Does the packing chromatic number grow when we take an infinite graph G and we glue the root of the one-way infinite path to a node in "a general position" (e.g. in the "middle" of G) ?
   $\boldsymbol{\chi_\rho(G + P_{\text{one way } \infty}) \geq \chi_\rho(G) + 1}$
5. Does 4 imply 1?

Merci

Slides: `http://kirgizov.link/talks/jga-2016.pdf`