Qubonacci words

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Definition

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factors of the form 0^a1^b satisfies $a = 0$ or $q \cdot a > b$.

$$\cdots 1\underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b}0\cdots$$

https://arxiv.org/abs/2010.09505

Max. factors 0^a1^b satisfies $a = 0$ or $q \cdot a > b$ for $q = 1$

$$\cdots 1\underbrace{000\cdots\cdots 00}_{a}\underbrace{111\cdots 11}_{b}0\cdots \qquad a > b$$

Let's count!

n	1	2	3	4	$\cdots$
	2	3	5	8	Fibonacci

0	00	000	0000	…
1	10	001	0001	
	11	100	0010	
		110	1000	
		111	1001	
			1100	
			1110	
			1111	

Max. factors 0^a1^b satisfies $a = 0$ or $q \cdot a > b$ for $q = 2$

$$\cdots 1 \underbrace{000 \cdots\cdots 00}_{a} \underbrace{111 \cdots 11}_{b} 0 \cdots \qquad \text{where } 2 \cdot a > b$$

Let's count!

n	1	2	3	4	...
	2	4	7	13	Tribonacci

0	00	000	0000	...
1	01	001	0001	
	10	010	0010	
	11	100	0011	
		101	0100	
		110	0101	
		111	1000	
			1001	
			1010	
			1100	
			1101	
			1110	
			1111	

Classical Fibonacci words

Binary words containing no occurrences of factor 1^k are enumerated by generalized Fibonacci numbers.

- Avoiding 11 : Fibonacci
- Avoiding 111 : Tribonacci
- etc

Donald Knuth. "The Art of Computer Programming, Volume 3"
2nd ed., page 286

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k-Gray code is a list of words where two consecutive words differ at k positions at most.

1-Gray code for these words is known.

- Matthew B. Squire, “Gray codes for A-free strings”, 1996
- Vincent Vajnovszki “A loopless generation of bitstrings without p consecutive ones”, 2001

Bijection with classical Fibonacci words

First define a map $\psi : \{0,1\}^n \to \{0,1\}^{n+q+1}$.

$$\psi(w) = \begin{cases} v001^{k+q} & \text{if } w = v01^k, k \geq 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Now, construct a length-preserving bijection ϕ that maps binary words avoiding 1^{q+1} to q-decreasing words.

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0, q], \\ \psi\big(\phi(v)\big) & \text{if } w = 1^q0v, \\ \phi(v)01^k & \text{if } w = 1^k0v \text{ and } k \in [0, q-1]. \end{cases}$$

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This is an example of what I would call a *wonderland* bijection. The mirror of any word avoiding 1^{q+1} also avoids 1^{q+1}. But a mirror of q-decreasing word is not necessary q-decreasing.

$av.111$	$\xrightarrow{\phi}$	2-dec.
1100		0011
1101		1111
1001		1001
1000		0001
1010		0101
1011		1101
0011		1100
0010		0100
0000		0000
0001		1000
0101		1010
0100		0010
0110		1110

The bijection does not preserve *Graycodeness*.

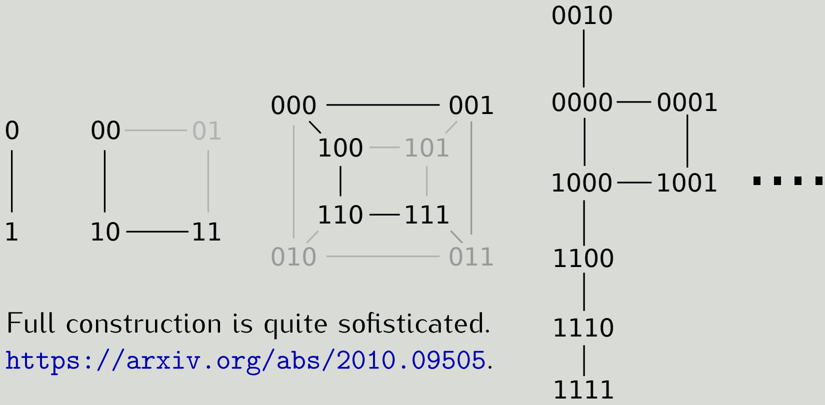
3-Gray code in general case and efficient generation

- Using Vajnovszki's lemma about *absorbent sets* from "Gray code order for Lyndon words" (2007) we prove that the set of n-length q-decreasing words admits a 3-Gray code.

- Efficient generation of q-decreasing words in lexicographical (or in 3-Gray code) order, $O(1)$ per word, is possible. It satisfies Frank Ruskey's constant amortized time (CAT) principle.

1-Gray code and solved conjecture

For $q = 1$ we managed to construct a 1-Gray code.



Full construction is quite sofisticated.
https://arxiv.org/abs/2010.09505.

We solve the conjecture about the existence of a Hamiltonian path in Fibonacci-run graphs, see Ömer Eğecioğlu and Vesna Iršič, "Fibonacci-run graphs I: Basic properties", 2021.

Enumeration

The bivariate generating function $W_q(x, y) = \sum_{n,k \geq 0} w_{n,k} x^n y^k$ where the coefficient $w_{n,k}$ is the number of n-length q-decreasing words containing exactly k 1s is given by:

$$W_q(x, y) = \frac{1 - x^{q+1} y^q}{1 - xy - x + x^{q+2} y^{q+1}}.$$

Mean bit

- In genereal we have more 1s in q-decreasing words than in words avoiding 1^{q+1}. Example:

	1-decreasing	avoiding 11
	000	000
	001	001
	100	010
	110	100
	111	101
Mean bit value	7/15	5/15

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	1-decreasing	avoiding 11
	000	000
	001	001
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	111	101
Mean bit value	7/15	5/15

- However, the mean bit values have common limit

$$\left.\frac{kx^k - kx^{k-1} - x^k + 1}{kx^k - kx^{k-1} + x^{2k} - 3x^k + 2}\right|_{x=1/\varphi_k} \quad \text{when } n \to \infty,$$

where φ_k is the generalized golden ratio, φ_2 is the golden ratio, and $k = q + 1$.

New paper in preparation...

Open questions

- We conjecture the existence of 1-Gray code for $q \geq 2, q \in \mathbb{N}$
- Investigate the case $q \in \mathbb{Q}^+$
- Other examples of wonderland bijections?

Wonderland bijection transforms a mirror-closed set to a set lacking this mirror symmetry.